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Sparseness of nonlinear coupling: importance in sparse direct-interaction perturbation

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Abstract

It is not weakness of nonlinearity but sparseness of nonlinear couplings that plays a key role in the sparse direct-interaction perturbation (SDIP), which is a second-order moment closure theory for nonlinear dynamical systems. The SDIP is similar but different in procedure from Kraichnan's direct-interaction approximation (DIA). Homogeneous Navier–Stokes turbulence is an example of dynamical systems in which nonlinearity is strong in magnitude but sparse in coupling. In order to clarify the importance of sparseness of coupling, we formulate SDIP for a model equation which has three parameters: coupling density, strength of nonlinearity and the number of degrees of freedom. By the help of numerical simulations, it is shown that SDIP is applicable when the coupling density is sufficiently smaller than the square root of the number of degrees of freedom, even if the strength of nonlinearity is infinitely large. This implies that the applicability of SDIP has nothing to do with the Gaussianity of a dynamical variable, whereas DIA is often explained as a theory based on it. The SDIP is also applied to a shell model of turbulence with sparse coupling which exhibits the Kolmogorov similarity. The solution to SDIP equations is consistent with inertial range properties such as the $-\frac{5}{3}$ power law of the energy spectrum.

Mathematics Subject Classification: 76F20, 37M99, 76F55

1. Introduction

Nonlinearity of a dynamical system is characterized by two parameters: strength and coupling density. The former represents the ratio of magnitudes of nonlinear and linear terms, which corresponds to the Reynolds number in the Navier–Stokes system. The latter is defined by the number of direct interactions between two modes. Let us consider the properties of these two parameters in homogeneous fluid turbulence described by the Navier–Stokes equation,

$$\frac{\partial}{\partial t}u_i(\boldsymbol{k},t) = \sum_{j=1}^{3}\sum_{m=1}^{3} M_{ijm}(\boldsymbol{k}) \sum_{\substack{\boldsymbol{p} \\ (\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{o})}}\sum_{\boldsymbol{q}} u_j(-\boldsymbol{p},t)u_m(-\boldsymbol{q},t) - \nu k^2 u_i(\boldsymbol{k},t), \tag{1}$$

where $u_i(\boldsymbol{k},t)$ is the Fourier component of velocity, $\boldsymbol{k}$ is the wavenumber, and

$$M_{ijm}(\boldsymbol{k}) = -\frac{\mathrm{i}}{2}\left(\frac{2\pi}{L}\right)^3 \left(k_m\delta_{ij} + k_j\delta_{im} - \frac{2k_ik_jk_m}{k^2}\right). \tag{2}$$

The flow is assumed to be periodic in three orthogonal directions with period L (see chapter V of [1] for derivation). The Reynolds number is defined by the ratio of magnitude of nonlinear and linear dissipative terms on the right-hand side of (1), which is much larger than unity in fully developed turbulence. The coupling density is the number of direct interactions between two modes, say $u_i(\boldsymbol{k}_1)$ and $u_i(\boldsymbol{k}_2)$, and is exactly equal to unity. To observe this, direct interactions between $u_i(\boldsymbol{k}_1)$ and other modes are schematically depicted in figure 1. There is only a single direct interaction between any pair of modes owning to the constraint $\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{o}$ in summations with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$. Since there are $\mathrm{O}(N)$, at the maximum, direct interactions between a pair of modes in an N mode quadratic nonlinear dynamical system, and since the number N of active modes is extremely large in fully developed turbulence, homogeneous turbulence at large Reynolds number may be regarded as a dynamical system of strong nonlinearity with sparse coupling. Incidentally, the sparseness introduced here has nothing to do with the sparse Fourier components of spatially symmetric flows such as the Taylor–Green vortex [2] and the high-symmetric flow [3] which were employed to save computer memory in direct numerical simulations of the Navier–Stokes equation.

A nonlinear term causes an infinite hierarchy of moment equations. Closure theories of nonlinear dynamical systems are after all to truncate this hierarchy, and to obtain a closed set of equations for a finite number of moments. Traditional perturbation theories based on the Reynolds number expansions are not valid for a system at large Reynolds numbers.

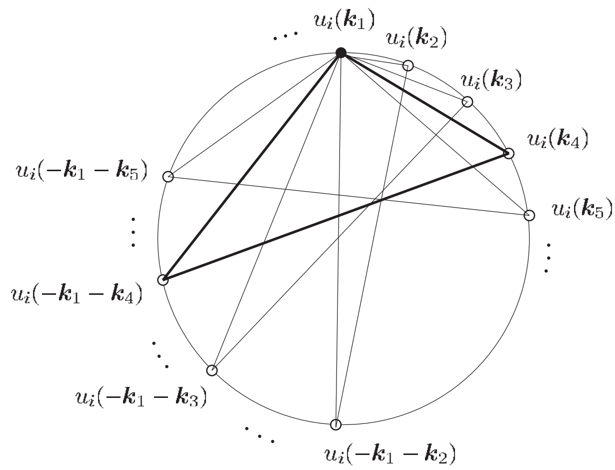

Figure 1. Schematic illustration of direct interactions between a Fourier mode $u_i(\boldsymbol{k}_1)$ and the others in the Navier–Stokes system. Each interaction is depicted by a triangle.

Fortunately, however, if nonlinear couplings are sparse as for the Navier–Stokes equation (1), there is a possibility to develop a successful closure. In this paper, we discuss the sparse direct-interaction perturbation (SDIP)[1] [9, 10], which is based on sparseness of coupling. As will be shown later, the coupling density should be compared with the number of degrees of freedom. It is usual that any approximation may require at least one small parameter. One of the purposes of this paper is to find such small parameters for validity condition of SDIP which depends on the coupling density and the number of degrees of freedom, and which can be small even in the large Reynolds number limit.

The rest of this paper is organized as follows. In the next section, we introduce a model equation, in which the strength of nonlinearity, the coupling density and the number of degrees of freedom are easily controlled. The SDIP is formulated for this model in section 3. It is shown numerically in section 4 how the accuracy of the prediction by SDIP is improved as the sparseness is decreased. Numerical evidences are given in section 5 for irrelativeness of Gaussianity of dynamical variables to the basis of SDIP, whereas the procedure of DIA is often explained based on the Gaussianity. The SDIP is successfully applied also to a shell model of turbulence in section 6. Concluding remarks are given in the last section.

2. Model equation and coupling density

By introducing a model equation [10–12] for a set of real variables $\{X_i \mid i = 1, 2, \ldots, N\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} X_i(t) = \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk} X_j(t) X_k(t) - \nu X_i(t) + f_i(t), \tag{3}$$

the coupling density of which is easily controlled. We shall discuss the essence of SDIP in detail. Real constant coefficients C_{ijk} are assumed to satisfy

$$\text{symmetry:} \qquad C_{ijk} = C_{ikj}, \tag{4}$$

$$\text{detailed balance of energy:} \qquad C_{ijk} + C_{jki} + C_{kij} = 0, \tag{5}$$

$$\text{absence of self-interaction:} \qquad C_{ijj} = 0, \tag{6}$$

$$\text{uniformity:} \qquad C_{ijk} = C_{\mathrm{rem}\{i+m,N\}\mathrm{rem}\{j+m,N\}\mathrm{rem}\{k+m,N\}} \quad \text{for } m = 1, 2, \ldots, N-1, \tag{7}$$

where rem denotes the remainder between 1 and N. Because of symmetric appearance of $X_j(t)$ and $X_k(t)$ in the nonlinear term, the anti-symmetric part of C_{ijk} is irrelevant so that we can always assume the symmetry condition (4). The detailed balance of energy (5) and the absence of self-interaction (6) are analogue of the Navier–Stokes equation (1). The condition (7) is imposed for simplicity since it is not essential to the present theme. Because of the uniformity of C_{ijk}, ν and f_i (see (8), below), statistics of $X_i(t)$ can be independent of i. This enables us to concentrate on investigation of a crucial aspect of SDIP, i.e. the importance of nonlinear coupling density. (See section 7 for another crucial aspect of SDIP in application to the Navier–Stokes turbulence—the introduction of Lagrangian velocity field.) The uniformity

[1] In order to avoid a confusion, we have renamed our theory from 'direct-interaction approximation' (DIA) to the SDIP. The name DIA is now permanently attached to Kraichnan's theories irrespective of any version [4–8]. The reason we used the same name in the past is that the results of our theory and his theory coincide in some special cases though the approximation procedures are different in an essential way. For example, in the SDIP procedure the deviation field is expressed by only three components of the zeroth-order field and the response function as will be seen in (B4), while in the commonly employed DIA procedure, which was called RRE in [10], it is expressed by full components of them. Compare (20) for SDIP and (47) for DIA (RRE version) in [10]. This difference is a consequence of the fact that only a single direct interaction is treated as a perturbation in SDIP, although in DIA (RRE version) full nonlinear term is done so.

condition (7) is not satisfied for the Navier–Stokes equation, but a nonuniform system will be also considered in section 6. A stepwise random force $f_i(t)$, which is constant during a time interval Δt, is assumed to obey a normal distribution with zero mean and variance,

$$\sigma^2 = \frac{2\,\nu}{N\,\Delta t}. \tag{8}$$

Instantaneous values of forcing terms, $f_i(t)$ and $f_j(t')$, are independent of each other either if $i \neq j$ or $|t - t'| > \Delta t$.

We define the coupling density ρ quantitatively by the average number of direct interactions between pairs of modes. In an N mode system, ρ can take a value between 0 and $N-2$. Only two extreme cases were dealt with in [10], namely the cases either that there is a single, at the most, direct interaction between any pair of modes ($\rho \sim 1$), or that all pairs of modes have $N-2$ direct interactions through all of the other modes ($\rho = N-2$). Here, we shall examine such systems that have intermediate coupling density. Incidentally, as stated in section 1, ρ is exactly equal to unity in the Navier–Stokes system (1). See appendix A for an example of a dynamical system with intermediate coupling density. Direct interactions between $X_1(t)$ and the other modes in several cases of coupling density are schematically depicted in figure 2 for $N = 20$.

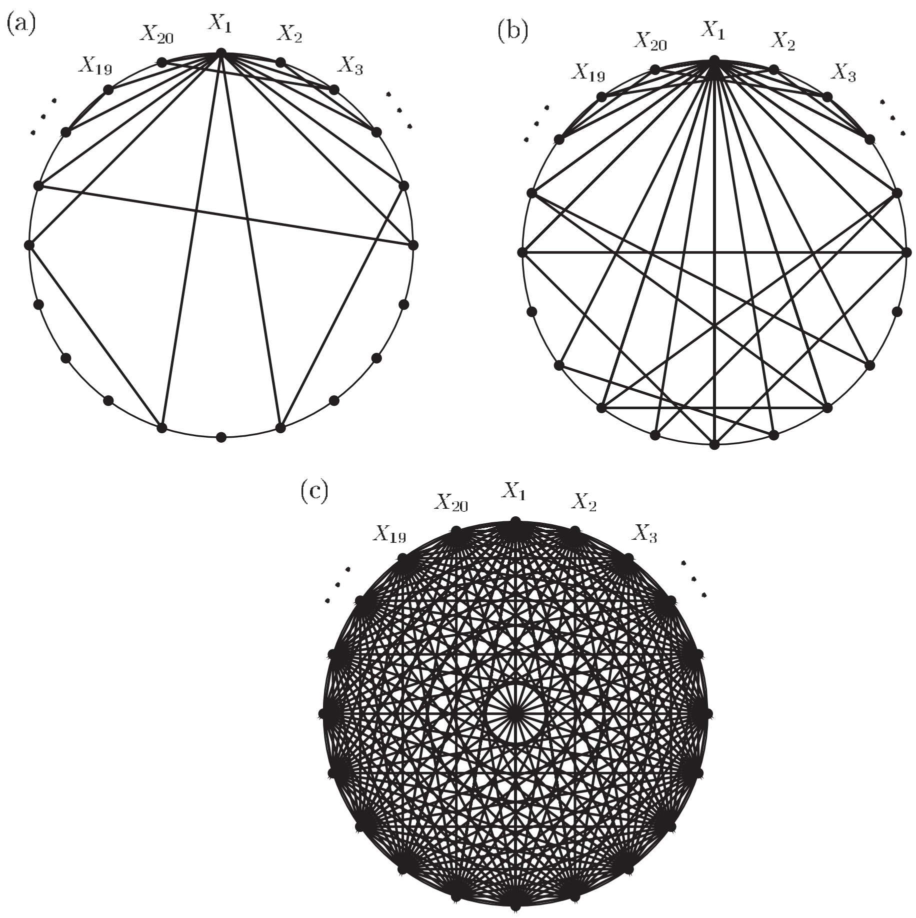

Figure 2. Direct interactions between X_1 and other modes in the model system (3). The coupling density is (*a*) $\rho \sim 1$, (*b*) 2 and (*c*) 18. $N = 20$.

Note that because of restrictions (4)–(7) of coefficients C_{ijk}, we cannot construct in general the model system (3) with ρ exactly equal to an integer except the densest coupling case. Actually we construct the coefficients C_{ijk} in such a way that the number of direct interactions between any pair of modes does not exceed $[\rho]+1$ in order to make the nonlinear couplings as uniform as possible.

The model equation (3) has three parameters: the dissipation coefficient ν (or the Reynolds number $R = 1/\nu$), the number N of degrees of freedom and the coupling density ρ. In the following sections we shall investigate how validity of SDIP depends on these parameters.

3. Sparse direct-interaction perturbation

3.1. Correlation and response functions

To start with, we separate the variable $X_i(t)$ into two parts:

$$X_i(t) = x_i(t) + Y_i(t), \tag{9}$$

where

$$x_i(t) = \overline{X_i(t)} \tag{10}$$

is an ensemble average (or a temporal average in statistically stationary cases), and $Y_i(t)$ is a fluctuation which satisfies

$$\overline{Y_i(t)} = 0. \tag{11}$$

It is easy to derive the evolution equation for $Y_i(t)$, from (3), (9)–(11), as

$$\frac{\mathrm{d}}{\mathrm{d}t}Y_i(t) = \sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}\left\{2\,x_j(t)Y_k(t) - \overline{Y_j(t)Y_k(t)} + Y_j(t)Y_k(t)\right\} - \nu\,Y_i(t) + f_i(t). \tag{12}$$

The purpose of second-order moment closure is to derive a closed set of equations for the correlation function of the fluctuation,

$$V(t,t') = \overline{Y_i(t)Y_i(t')}, \tag{13}$$

which is governed by

$$\left(\frac{\partial}{\partial t} + \nu\right)V(t,t') = \sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}\left\{2\,x_j(t)\overline{Y_k(t)Y_i(t')} + \overline{Y_j(t)Y_k(t)Y_i(t')}\right\}, \qquad t > t' \tag{14}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + 2\nu\right)V(t,t) = 2\sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}\left\{2\,x_j(t)\overline{Y_k(t)Y_i(t)} + \overline{Y_j(t)Y_k(t)Y_i(t)}\right\} + 2\,\overline{f_i(t)Y_i(t)}, \tag{15}$$

and for a few other statistical quantities (if necessary). In SDIP we introduce the response function,

$$G_{in}(t,t') = \frac{\delta Y_i(t)}{\delta Y_n(t')}, \tag{16}$$

where δ denotes a functional derivative and $G_{in}(t,t) = \delta_{in}$. The evolution equation of $G_{in}(t,t')$ is obtained by taking the functional derivative of (3) with respect to $Y_n(t')$ as

$$\frac{\partial}{\partial t}G_{in}(t,t') = 2\sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}\{x_j(t) + Y_j(t)\}G_{kn}(t,t') - \nu\,G_{in}(t,t'), \qquad t > t', \tag{17}$$

an ensemble average of which leads to

$$\frac{\partial}{\partial t}\overline{G_{in}(t,t')} = 2\sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}\left\{x_j(t)\overline{G_{kn}(t,t')} + \overline{Y_j(t)G_{kn}(t,t')}\right\} - \nu\,\overline{G_{in}(t,t')}, \qquad t > t', \tag{18}$$

with initial condition $\overline{G_{in}(t,t)} = \delta_{in}$. The higher-order statistical quantities appearing on the right-hand sides of (14), (15) and (18) must be expressed in terms of the quantities on the left-hand sides under some reasonable procedures. In the following subsections, we perform it by SDIP [9, 10].

3.2. Formulation of SDIP

The SDIP is based on the sparseness of nonlinear couplings, and consists of the following two assumptions, which have been justified numerically [10]. If we artificially remove a single direct interaction between $Y_{i_0}(t)$, $Y_{j_0}(t)$ and $Y_{k_0}(t)$, say, then (I) these three modes are statistically independent of each other, but (II) statistical properties of the entire system are hardly changed. In order to put these assumptions into practice, we introduce a decomposition (the direct-interaction decomposition) at $t = t_0$ that

$$Y_i(t) = Y^{(0)}_{i/i_0 j_0 k_0}(t|t_0) + Y^{(1)}_{i/i_0 j_0 k_0}(t|t_0). \tag{19}$$

Here, an artificial field $Y^{(0)}_{i/i_0 j_0 k_0}(t|t_0)$, called the non-direct-interaction field associated with a triplet of modes $\{Y_{i_0}, Y_{j_0}, Y_{k_0}\}$, is governed by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y^{(0)}_{i/i_0 j_0 k_0}(t|t_0) &= \sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}\left\{2\,x_j(t)Y^{(0)}_{k/i_0 j_0 k_0}(t|t_0) - \overline{Y_j(t)Y_k(t)}\right\}\\ &+ \sum_{\substack{j=1\\ \{i,j,k\}\neq\{i_0,j_0,k_0\}}}^{N}\sum_{k=1}^{N} C_{ijk}Y^{(0)}_{j/i_0 j_0 k_0}(t|t_0)Y^{(0)}_{k/i_0 j_0 k_0}(t|t_0) - \nu\,Y^{(0)}_{i/i_0 j_0 k_0}(t|t_0) + f_i(t),\end{aligned} \tag{20}$$

and a deviation field $Y^{(1)}_{i/i_0 j_0 k_0}(t|t_0)$ by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Y^{(1)}_{i/i_0 j_0 k_0}(t|t_0) &= \sum_{j=1}^{N}\sum_{k=1}^{N} 2C_{ijk}x_j(t)Y^{(1)}_{k/i_0 j_0 k_0}(t|t_0)\\ &+ \sum_{\substack{j=1\\ \{i,j,k\}\neq\{i_0,j_0,k_0\}}}^{N}\sum_{k=1}^{N} 2C_{ijk}Y_j(t)Y^{(1)}_{k/i_0 j_0 k_0}(t|t_0) - \nu\,Y^{(1)}_{i/i_0 j_0 k_0}(t|t_0)\\ &+\delta_{ii_0}2C_{i_0 j_0 k_0}Y^{(0)}_{j_0/i_0 j_0 k_0}(t|t_0)Y^{(0)}_{k_0/i_0 j_0 k_0}(t|t_0)\\ &+\delta_{ij_0}2C_{j_0 k_0 i_0}Y^{(0)}_{k_0/i_0 j_0 k_0}(t|t_0)Y^{(0)}_{i_0/i_0 j_0 k_0}(t|t_0)\\ &+\delta_{ik_0}2C_{k_0 i_0 j_0}Y^{(0)}_{i_0/i_0 j_0 k_0}(t|t_0)Y^{(0)}_{j_0/i_0 j_0 k_0}(t|t_0).\end{aligned} \tag{21}$$

Observe that a difference between the governing equations for $Y_i(t)$ and $Y^{(0)}_{i/i_0 j_0 k_0}(t|t_0)$ is only in the absence of a single direct interaction between $\{Y_{i_0}, Y_{j_0}, Y_{k_0}\}$ in the latter. Then, the SDIP assumptions are stated as follows.

- *SDIP assumption 1.* Three modes $\{Y^{(0)}_{i_0/i_0 j_0 k_0}, Y^{(0)}_{j_0/i_0 j_0 k_0}, Y^{(0)}_{k_0/i_0 j_0 k_0}\}$, between which the direct interaction is absent, are statistically independent of each other.

- *SDIP assumption 2.* The deviation field $Y^{(1)}_{i/i_0j_0k_0}(t|t_0)$ is much smaller in magnitude than the true field $Y_i(t)$ in a statistical sense as long as $t - t_0$ is within the timescale of the auto-correlation function $V(t, t')$.

Similar assumptions are imposed for the response function (see [10] for detail). Hereafter, we omit the argument t_0 of $Y^{(0)}_{i/i_0j_0k_0}(t|t_0)$ for brevity of notations.

These two assumptions are satisfied better and better as the number N of degrees of freedom of the system increases. As for the first assumption, when N is large, many modes are coupled indirectly with those particular three modes, and such indirect interactions may be well randomized and tend to cancel out correlations between them in a statistical sense. As for the second assumption, an artificial removal of one direct interaction may have only tiny effects on the entire statistics of a large system. Furthermore, it should be stressed here that the first assumption requires also sparseness of nonlinear couplings. If couplings are denser, there exist more indirect interactions between a triplet of modes through a small number of modes (e.g. four-mode indirect interactions shown in figure 3), which are not expected to be randomized enough to make correlations between the three modes negligible. Indeed, it is shown [10] that in the densest coupling system ($\rho = N - 2$) this assumption is violated, and the prediction of auto-correlation function by SDIP is far from satisfaction. Density dependence of accuracy of prediction by SDIP will be estimated quantitatively in section 4.

The SDIP assumption 1 is employed as

$$\overline{Y^{(0)}_{i/ijk}Y^{(0)}_{j/ijk}Y^{(0)}_{k/ijk}} = 0 \tag{22}$$

and

$$\overline{Y^{(0)}_{i/ijk}Y^{(0)}_{i/ijk}Y^{(0)}_{j/ijk}Y^{(0)}_{j/ijk}} = \overline{Y^{(0)}_{i/ijk}Y^{(0)}_{i/ijk}}\;\overline{Y^{(0)}_{j/ijk}Y^{(0)}_{j/ijk}} \tag{23}$$

in the SDIP formulation (see appendix B). It is a guiding principle in moment closure theories to express higher-order moments in terms of lower-order ones. By a straightforward calculation, using (22) and (23), we can derive from (14), (15) and (18) a closed set of equations for the two-mode correlation and the response functions as

$$\left(\frac{\partial}{\partial t} + \nu\right)V(t,t') = -2\mathcal{C}\int_{t_0}^{t} \mathrm{d}t''\, G(t,t'')V(t',t'')V(t,t'') + 2\mathcal{C}\int_{t_0}^{t'} \mathrm{d}t''\, G(t',t'')V(t,t'')^2, \qquad t > t', \tag{24}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + 2\nu\right)V(t,t) = \frac{2\nu}{N} \tag{25}$$

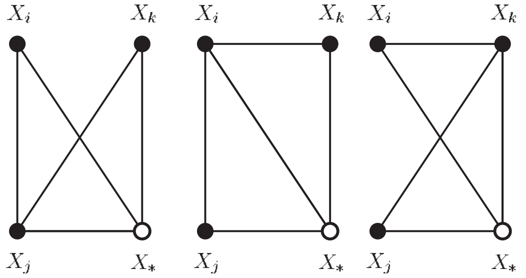

Figure 3. Four-mode indirect interactions between X_i, X_j and X_k through a mode X_*.

and

$$\left(\frac{\partial}{\partial t}+\nu\right)G(t,t') = -2\,\mathcal{C}\int_{t'}^{t} \mathrm{d}t''\,V(t,t'')G(t,t'')G(t'',t'). \tag{26}$$

Here, $G(t,t')$ and $\mathcal{C}$ are defined by

$$G(t,t') = \overline{G_{ii}(t,t')} \tag{27}$$

and

$$\mathcal{C} = \sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk}^2, \tag{28}$$

respectively. The linear equation (25) can be integrated to be

$$V(t,t) = \frac{1}{N} + \left\{V(0,0) - \frac{1}{N}\right\}\mathrm{e}^{-2\nu t}. \tag{29}$$

It is interesting to see that (24) and (26) permit a solution which satisfies

$$V(t,t') = V(t',t')\,G(t,t'), \tag{30}$$

since each of them then leads to

$$\left(\frac{\partial}{\partial t}+\nu\right)G(t,t') = -2\,\mathcal{C}\int_{t'}^{t} \mathrm{d}t''\,G(t,t'')G(t,t'')G(t'',t')V(t'',t''). \tag{31}$$

Equation (30) implies that the response function plays the role of a linear propagator on the correlation function. It should be noted that the artificial time t_0 in (24) disappears automatically as soon as (30) is assumed. Hereafter, we call (29)–(31) the SDIP equations.

4. Accuracy of SDIP

Here, we show explicitly that the SDIP equations give an extremely good prediction even in the limit of the large Reynolds numbers ($R \to \infty$) by examining the solutions in the statistically stationary state, in which V and G may be expressed as

$$\begin{aligned} \mathcal{V}(\tau) &= V(t+\tau, t), \\ \mathcal{G}(\tau) &= G(t+\tau, t), \end{aligned} \tag{32}$$

where $\tau \geqslant 0$. Then the SDIP equations (29)–(31) are reduced to

$$\mathcal{V}(0) = \frac{1}{N}, \tag{33}$$

$$\mathcal{V}(\tau) = \mathcal{V}(0)\,\mathcal{G}(\tau) \tag{34}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+\nu\right)\mathcal{V}(\tau) = -\frac{2\,\mathcal{C}}{\mathcal{V}(0)}\int_0^{\tau} \mathrm{d}\tau'\,\mathcal{V}(\tau')^2\,\mathcal{V}(\tau-\tau'). \tag{35}$$

For later use, we define here the auto-correlation time τ_{c} as

$$\tau_{\mathrm{c}}(\rho, N) = \frac{1}{\sqrt{2\,\mathcal{C}\,\mathcal{V}(0)}}. \tag{36}$$

The SDIP equation (35) does not depend on ρ and N in the limit $\nu \to 0$, if $\mathcal{V}$ and τ are, respectively, rescaled by $\mathcal{V}(0)$ and $\tau_{\mathrm{c}}(\rho, N)$.

Figure 4 compares predictions $\mathcal{V}_{\mathrm{SDIP}}$ of the auto-correlation function by the SDIP equations (35) and results $\mathcal{V}_{\mathrm{DNS}}$ of direct numerical simulations of (3) for several combinations

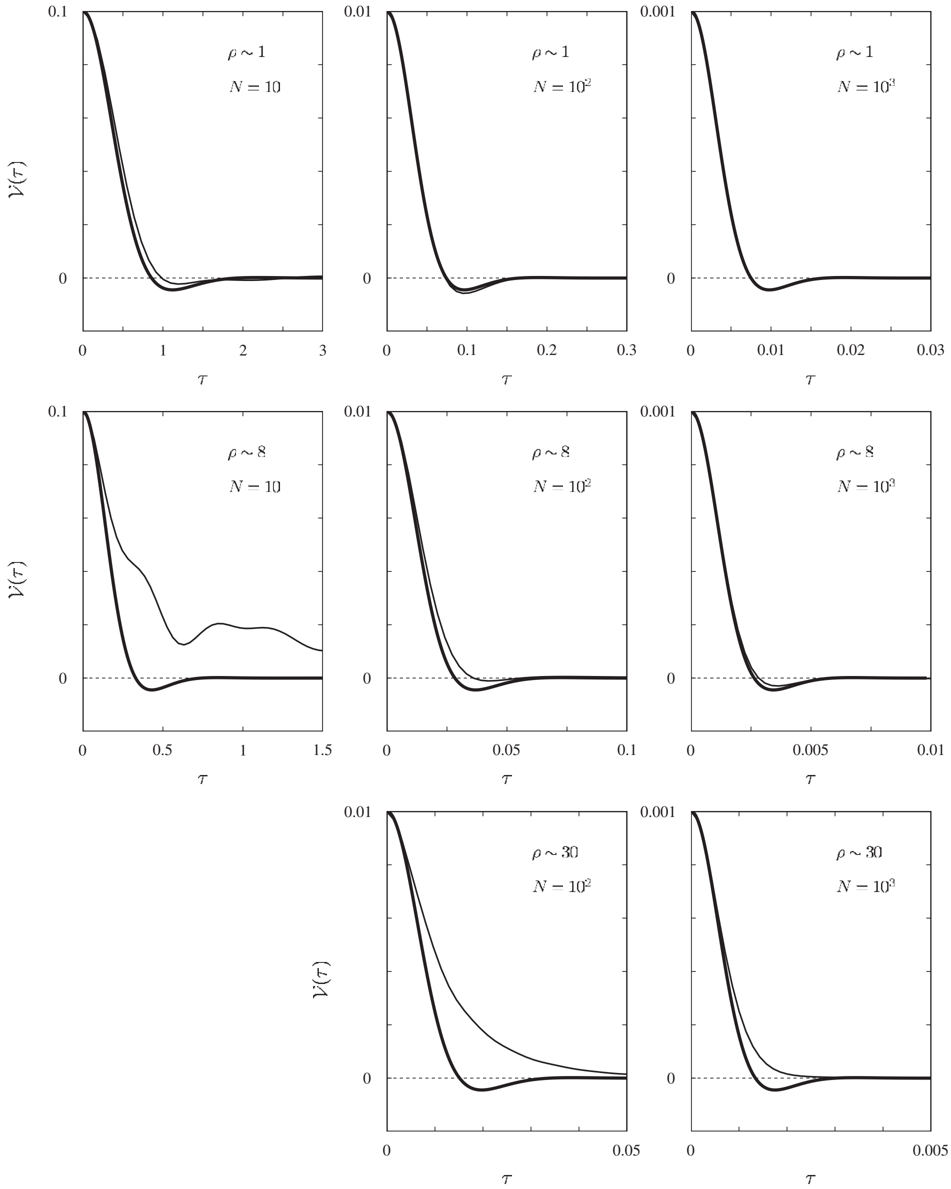

Figure 4. Auto-correlation function of the model equation (3) for various combinations of N and ρ in the case $R \to \infty$. (——): SDIP; (——): direct numerical simulation.

of (ρ, N) at the infinite Reynolds number (i.e. $\nu = 0$). The agreement is better for larger N or smaller ρ, and it is excellent in the case that $\rho \sim 1$ and $N \gg 1$. This is consistent with the intuitive argument made in section 3.2 that SDIP should be more accurate as the nonlinear couplings are sparser and the number of degrees of freedom is larger.

In order to make a quantitative comparison, we introduce an integrated difference,

$$\Delta_{\mathrm{SDIP}} = \frac{\int_0^\infty \mathrm{d}\tau\, |\mathcal{V}_{\mathrm{SDIP}}(\tau) - \mathcal{V}_{\mathrm{DNS}}(\tau)|}{\int_0^\infty \mathrm{d}\tau\, |\mathcal{V}_{\mathrm{DNS}}(\tau)|}, \tag{37}$$

which represents a discrepancy between a direct numerical simulation result and a prediction by the SDIP equations, i.e. small Δ_{SDIP} implies validness of the SDIP equations. Numerical results of Δ_{SDIP} evaluated for various ρ ($\ll N$) are plotted in figures 5(*a*) and (*b*) for $N = 10^2$ and $N = 10^3$, respectively, in the limit of $R \to \infty$. Their ρ dependence is qualitatively consistent with the above results on the auto-correlation function. As seen in figure 5(*c*),

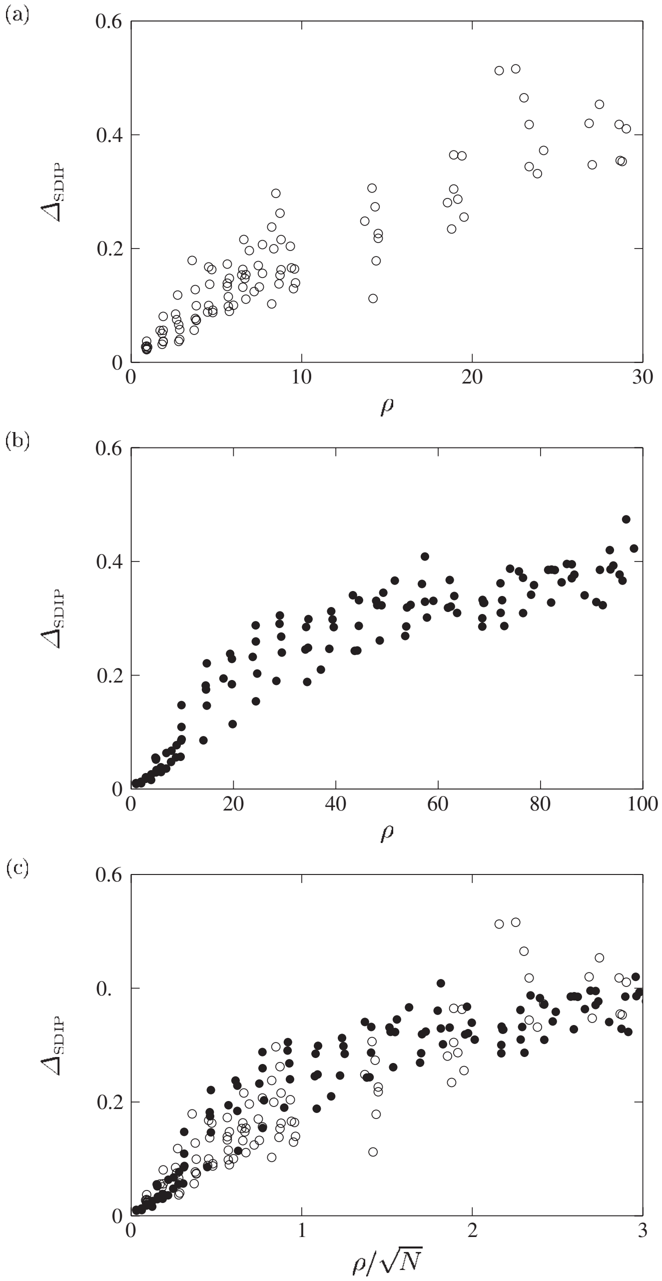

Figure 5. Coupling density dependence of the integrated difference Δ_{SDIP}. $\nu = 0$. (*a*) $N = 10^2$, (*b*) $N = 10^3$ and (*c*) (○): $N = 10^2$; (●): $N = 10^3$.

$\Delta_{\rm SDIP}$ seems to be a function of $\rho/\sqrt{N}$. Thus, it is numerically suggested that the SDIP equations may be valid as long as

$$\rho \ll \sqrt{N} \tag{38}$$

even if $R \to \infty$.

This validity condition (38) of the SDIP equations is consistent with SDIP assumption 1, which is grounded on sparseness of nonlinear couplings and on largeness of the number of degrees of freedom. This may be understood as follows. The simplest indirect interactions between a triplet of modes $\{X_i, X_j, X_k\}$ are four-mode interactions which involve one more mode X_* (figure 3). Since such a small-number-mode interaction may bring non-negligible correlation between the three modes, the independency assumption (SDIP assumption 1) deteriorates. (It requires that indirect interactions should be randomized enough so that their effects to the correlation can be neglected in a statistical sense.) The probability q of the existence of a four-mode interaction is evaluated as

$$q = \begin{cases} 0 & \rho \leqslant 1, \\ \dfrac{3\rho(\rho-1)}{N}, & \rho > 1, \end{cases} \tag{39}$$

for $N \gg 1$. It is expected that $\Delta_{\rm SDIP}$ is smaller for smaller q. This may be the reason why $\Delta_{\rm SDIP}$ is a function of $\rho/\sqrt{N}$; note that $q \approx 3(\rho/\sqrt{N})^2$ for $\rho \gg 1$ and $N \gg 1$. It is concluded, therefore, that SDIP should be valid for systems which satisfy the condition (38) irrespective of the Reynolds number.

5. Irrelativeness of SDIP to Gaussianity

It is known (see, for example, section 19.6 of [13]) that Kraichnan's DIA equations are derived by a variety of methods based on different ideas and procedures[2]. In contrast with our SDIP described in the preceding section, many of the authors (e.g. [11, 14, 15]) assumed the Gaussianity of dynamical variables, and employed the well-known mathematical properties of a set of probability variables, $\{Y_i^{(G)} | i = 1, 2, 3, \ldots\}$ which obeys a joint Gaussian distribution with zero mean: an odd-order moment vanishes, e.g.

$$\overline{Y_i^{(G)} Y_j^{(G)} Y_k^{(G)}} = 0, \tag{40}$$

and an even-order moment is expressed in terms of second-order ones, e.g.

$$\overline{Y_i^{(G)} Y_j^{(G)} Y_k^{(G)} Y_l^{(G)}} = \overline{Y_i^{(G)} Y_j^{(G)}}\;\overline{Y_k^{(G)} Y_l^{(G)}} + \overline{Y_i^{(G)} Y_l^{(G)}}\;\overline{Y_j^{(G)} Y_k^{(G)}} + \overline{Y_i^{(G)} Y_k^{(G)}}\;\overline{Y_j^{(G)} Y_l^{(G)}}. \tag{41}$$

It should be emphasized that (40) and (41) are formally resemblant to (22) and (23), but their bases are essentially different from each other in the sense that the validity of (22) and (23) has nothing to do with the Gaussianity of the dynamical variable. In the following we perform two kinds of examinations which show that SDIP is never based on Gaussianity. In the first examination SDIP does not work well, though the pdf of $X_i(t)$ is nearly Gaussian. In the second examination the pdf is far from Gaussian but the prediction by SDIP is excellent.

The mth order moment of $X_i(t)$ is defined by

$$M_m(t) = \overline{X_i(t)^m}. \tag{42}$$

[2] As stated at the footnote in the introduction, our SDIP also gives the same DIA equations in some special cases although not always.

Here, we deal with a non-stationary uniform system, in which an average is defined by

$$\bar{z}(t) = \frac{1}{NJ}\sum_{i=1}^{N}\sum_{j=1}^{J} z_i(t;\, j) \tag{43}$$

instead of the temporal average adopted in a stationary case. Index j denotes a realization of J ($\gg$1) initial conditions. The average (43) is taken over both N modes and J initial conditions. For the purpose of showing the irrelativeness of SDIP to Gaussianity, the following four initial pdfs $P(X_i, t = 0)$, which satisfy

$$M_2(0) = \frac{1}{N}, \tag{44}$$

are examined:

$$\text{Init-(a):} \qquad P(X_i, 0) = \frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left(-\frac{X_i^2}{2\sigma_1^2}\right), \tag{45}$$

$$\text{Init-(b):} \qquad P(X_i, 0) = \tfrac{1}{2}\left\{\delta\left(X_i - \tfrac{1}{2}\right) + \delta\left(X_i + \tfrac{1}{2}\right)\right\}, \tag{46}$$

$$\text{Init-(c):} \qquad P(X_i, 0) = \begin{cases} 0, & X_i \leqslant -\sigma_2^2, \\ \dfrac{1}{\sqrt{2\pi\sigma_2^2(X_i + \sigma_2^2)}}\exp\left(-\dfrac{X_i + \sigma_2^2}{2\sigma_2^2}\right), & X_i > -\sigma_2^2, \end{cases} \tag{47}$$

$$\text{Init-(d):} \qquad P(X_i, 0) = \begin{cases} 0, & X_i \leqslant 0, \\ \dfrac{1}{\sqrt{2\pi\sigma_3^2 X_i}}\exp\left(-\dfrac{X_i}{2\sigma_3^2}\right) & X_i > 0, \end{cases} \tag{48}$$

where

$$\sigma_1^2 = \frac{1}{N}, \qquad \sigma_2^2 = \left(\frac{1}{2N}\right)^{1/2}, \qquad \sigma_3^2 = \left(\frac{1}{3N}\right)^{1/2}. \tag{49}$$

Initial pdf (a) is Gaussian, whereas the other three are far from Gaussian. Initial pdf (b) has two peaks, (c) and (d) have non-vanishing skewness factors, and (d) has non-vanishing mean $M_1(0) = \sigma_3^2$ as well.

Temporal evolutions of the pdfs obtained by direct numerical simulations of (3) with $N = 10^3$, $\nu = 0$ and $\rho \sim 1$ are plotted in figure 6. It is observed that each system relaxes to a statistically stationary state, which does not depend on the initial conditions. This statistically stationary state may be Ergodic, in which a state point migrates densely and homogeneously on an equi-energy surface in the N-dimensional phase space. The pdf is then estimated as

$$P(X_i, \infty) = \frac{1}{\sqrt{\pi}}\frac{\Gamma(N/2)}{\Gamma((N-1)/2)}(1 - X_i^2)^{(N-3)/2}, \tag{50}$$

where Γ is the Gamma function. The numerically obtained pdf for various values of the coupling density ρ actually tends to (50) at $t \gg \tau_c$ (figure 7). Here, we should note that expression (50) is valid irrespective of coupling density ρ as long as a conservative system is concerned, and that (50) tends to a Gaussian in the limit of $N \gg 1$. These facts imply that SDIP, which strongly depends on coupling density, is not necessarily stuck with Gaussianity. In the statistically stationary state ($t \gg \tau_c$) of the system with *dense* couplings and large degrees of freedom, the pdf is nearly Gaussian but SDIP does not work since (38) is violated for dense ρ as seen in the preceding section. This is the first examination which shows that SDIP has no relation with Gaussianity.

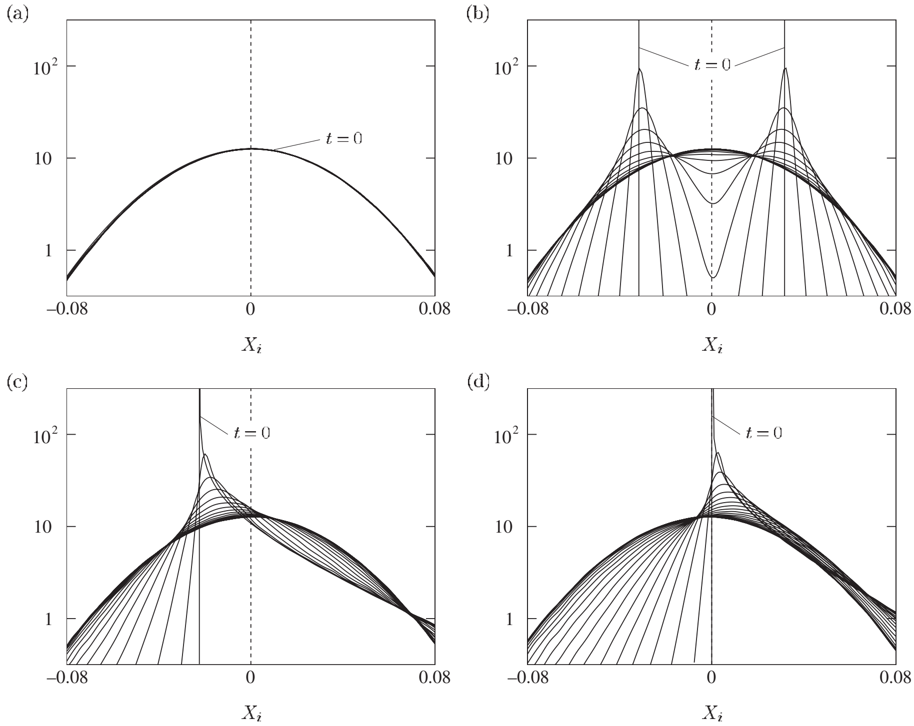

Figure 6. Temporal evolution of the pdf $P(X_i, t)$. Each line denotes $P(X_i, t)$; $t = 0, \Delta t, 2\Delta t, 3\Delta t, \ldots$ ($\Delta t \approx 0.1\tau_c$). Figures (*a*)–(*d*) correspond to the initial pdfs (*a*)–(*d*), respectively. $N = 10^3$, $\nu = 0$, $\rho \sim 1$, $J = 10^3$.

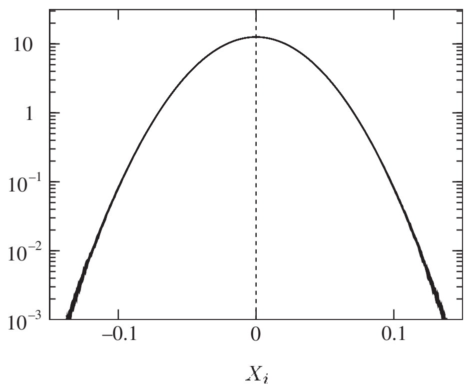

Figure 7. Pdf of $X_i(t)$ in the statistically stationary state ($t \gg \tau_c$) obtained by direct numerical simulations. Coupling density $\rho \approx 1, 10, 20, 30, \ldots, 100$; these curves overlap not only with each other but also with the theoretical prediction (50). $N = 10^3$, $\nu = 0$, $J = 10^3$.

In terms of the skewness factor,

$$s(t) = \frac{M_3(t)}{M_2(t)^{3/2}}, \tag{51}$$

and the flatness factor,

$$f(t) = \frac{M_4(t)}{M_2(t)^2}, \tag{52}$$

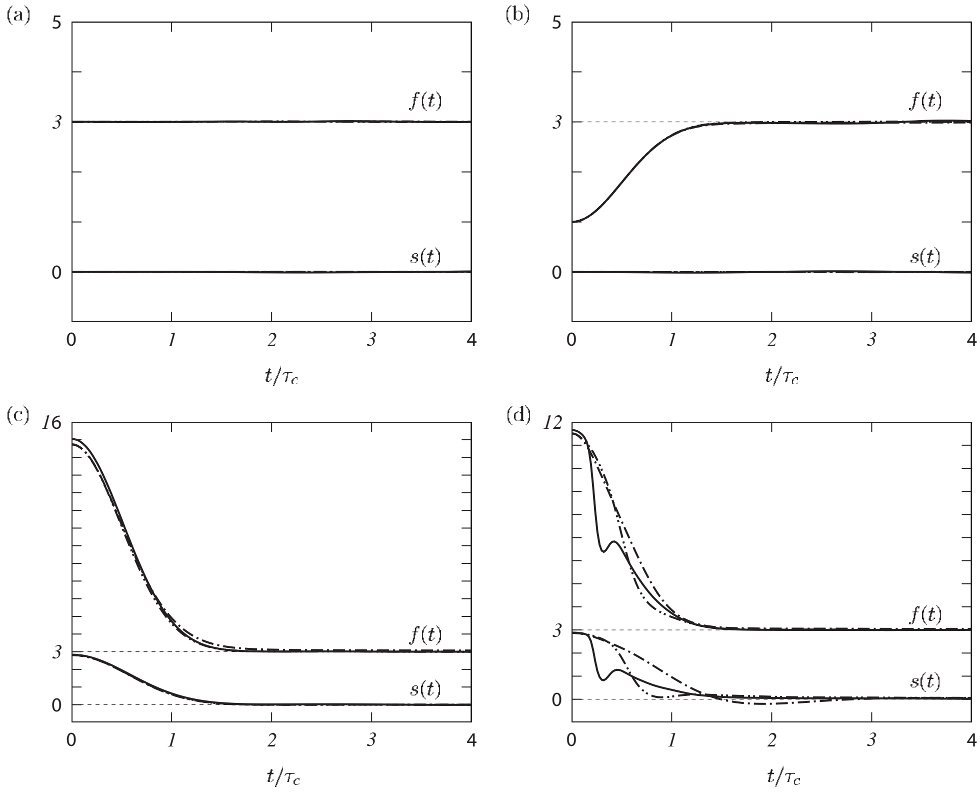

Figure 8. Temporal evolutions of the skewness $s(t)$ and the flatness factors $f(t)$ of $X_i(t)$. (— · —): $N = 10^3$; (— · · —): $N = 10^4$; (— · · · —): $N = 10^5$. Figures (*a*)–(*d*) correspond to initial pdfs (*a*)–(*d*). $\nu = 0$, $\rho \sim 1$. $J = 10^3, 10^2, 10$ for $N = 10^3, 10^4, 10^5$, respectively.

we may estimate the relaxation timescale from the initial non-Gaussian pdf to the one expressed by (50). Simulation results of the skewness and flatness factors are plotted in figure 8, in which time t is normalized by the auto-correlation time τ_c (see (36)). It is seen in this figure that the deviation of pdf $P(X_i, t)$ from a Gaussian is significant before the auto-correlation time ($t \lesssim \tau_c$) even if N is large when the initial pdf is non-Gaussian.

Now we proceed to investigate the auto-correlation function,

$$\tilde{\mathcal{V}}(t) = V(t, 0), \tag{53}$$

of a non-Gaussian variable when the pdf $P(X_i, 0)$ is taken as one of (b)–(d). It is easy to show from (29) to (31) that the SDIP equation for $\tilde{\mathcal{V}}(t)$ is the same as (35) in the limit of $\nu \to 0$. Figure 9 compares direct numerical simulation results of $\tilde{\mathcal{V}}(t)$ with the prediction by the SDIP equation (35) for each of the four initial pdfs. The SDIP predictions are excellent irrespective of the initial pdfs. This is the second examination which explicitly exhibits irrelativeness of SDIP and Gaussianity.

6. Application of SDIP to a shell model of turbulence

In previous sections, we dealt with a uniform system as a simple case. Here, we extend it to a non-uniform system which is closer to the Navier–Stokes equation, and show that SDIP works excellently.

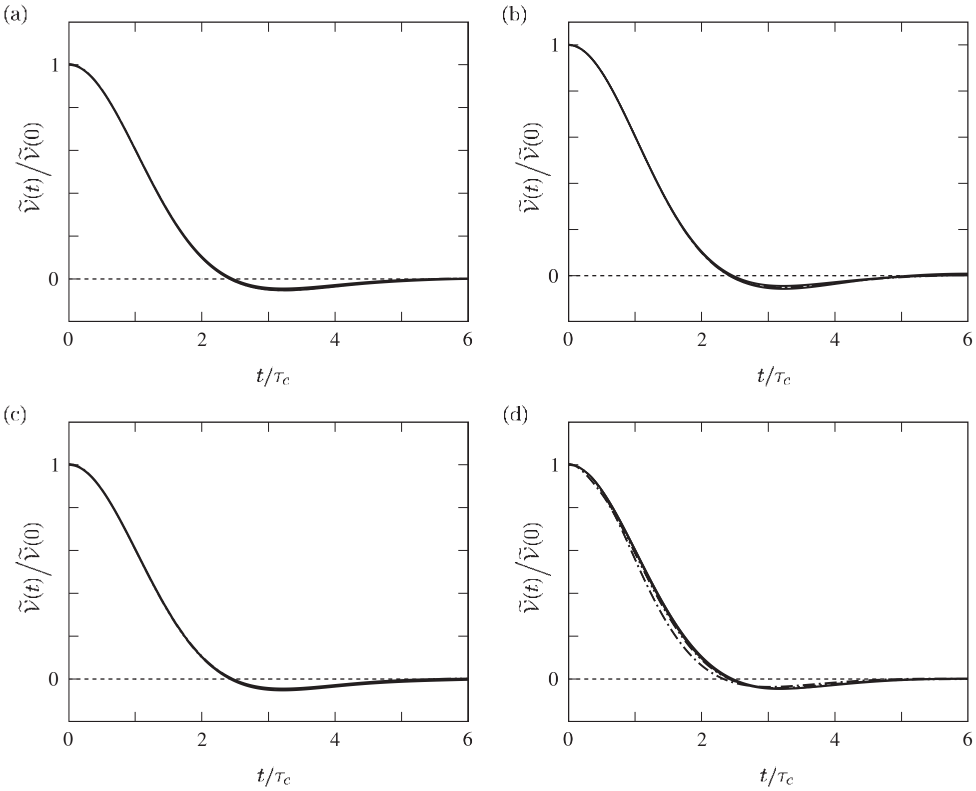

Figure 9. Auto-correlation functions $\tilde{\mathcal{V}}(t)$. In each figure, direct numerical simulation results for $N = 10^3$ (— · —), $N = 10^4$ (— · · —) and $N = 10^5$ (— · · · —) are plotted together with the solution (——) to the SDIP equation. Figures (*a*)–(*d*) correspond to initial pdfs (*a*)–(*d*). $\nu = 0$, $\rho \sim 1$. $J = 10^3, 10^2, 10$ for $N = 10^3, 10^4, 10^5$, respectively.

Let us consider a dynamical system of real variables $\{X_i^{(l)} \mid i = 1, 2, \ldots, N;\ l = 1, 2, \ldots, l_{\max}\}$ as

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} + \nu\left(k^{(l)}\right)^2\right\} X_i^{(l)} = 2\,k^{(l)} \sum_{j=1}^{N}\sum_{m=1}^{N} C_{ijm} X_j^{(l)}(t) X_m^{(l+1)}(t)$$
$$+k^{(l-1)} \sum_{j=1}^{N}\sum_{m=1}^{N} C_{ijm} X_j^{(l-1)}(t) X_m^{(l-1)}(t) + f\delta_{l1}. \tag{54}$$

Here, C_{ijm} is the same as the coefficient introduced in section 2 with $\rho \sim 1$, and $k^{(l)}$ is the wavenumber defined by

$$k^{(l)} = D^l. \tag{55}$$

System (54) may be regarded as a kind of shell models of turbulence, which is composed of $l_{\max}$ shells of N components (see [16] for a review of shell models). Any pair of modes $X_i^{(l)}$ and $X_j^{(l')}$ in this model has only a single direct interaction (i.e. sparse coupling) and the interactions are limited in adjacent shells. This latter property of localization of nonlinear interactions in the wavenumber space is an essence of shell models, and implies that the dynamical variable $X_i^{(l)}$ mimics the Fourier mode of the Lagrangian velocity. Note that the Eulerian velocity suffers from non-local-in-scale interactions known as the sweeping effects. It is interesting that this model, as well as other shell models, exhibits the Kolmogorov similarity [17]. Figure 10(a) plots the energy spectrum

$$E^{(l)}(t) = \frac{1}{2\,k^{(l)}} \sum_{i=1}^{N} \overline{(u_i^{(l)}(t))^2} \tag{56}$$

obtained by direct numerical simulation ($D = e^{1/2}$, $\nu = 3.0 \times 10^{-8}$, $f = 1$, $N = 10$ and $l_{\max} = 32$) of (54), where the wavenumber and the spectrum are normalized by $k_d = \epsilon^{3/4}\nu^{-3/4}$ and $\epsilon^{1/4}\nu^{5/4}$, respectively. Here, $\epsilon = 2\nu \sum_{l=1}^{l_{\max}} (k^{(l)})^3\, E^{(l)}$ is the energy dissipation rate. It is observed that $E^{(l)}$ obeys the $(k^{(l)})^{-5/3}$ power law in the inertial range ($k^{(l)} \ll k_d$).

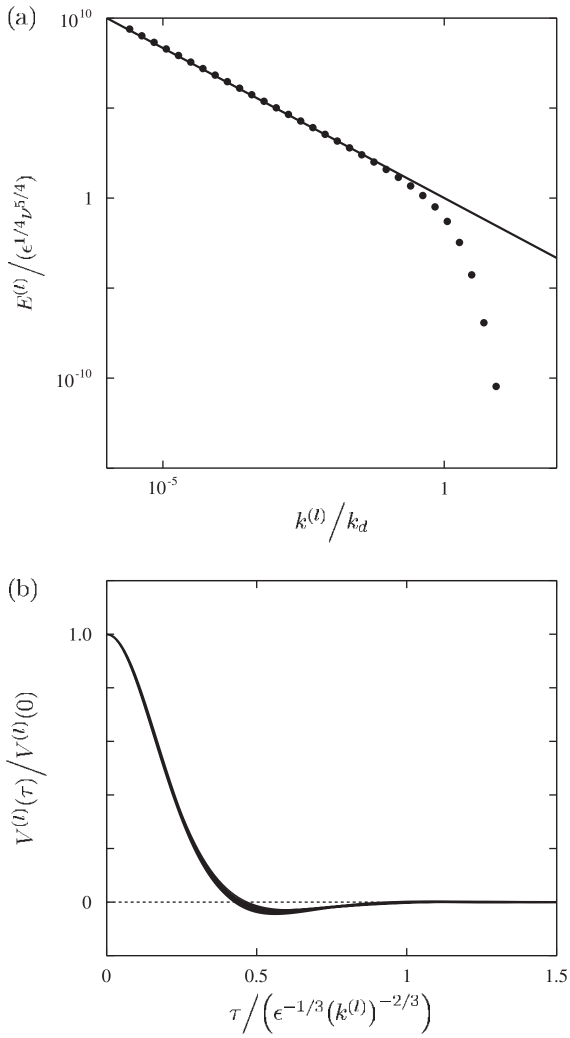

Figure 10. (a) Energy spectrum of the shell model. $D = e^{1/2}$, $\nu = 3.0 \times 10^{-8}$, $f = 1$, $N = 10$ and $l_{\max} = 32$. Solid line denotes the prediction by SDIP in the inertial range, $E^{(l)} = 1.01\epsilon^{2/3}(k^{(l)})^{-5/3}$. ($b$) Auto-correlation function in the inertial range. Thin 16 lines are results of direct numerical simulation, thick line is prediction by SDIP.

Procedure of SDIP for system (54) is essentially the same as that for (3) described in section 3.2. In terms of the auto-correlation function,

$$V^{(l)}(|t-t'|) = \overline{X_i^{(l)}(t)X_i^{(l)}(t')}, \tag{57}$$

and the averaged auto-response function,

$$G^{(l)}(t-t') = \overline{\frac{\delta X_i^{(l)}(t)}{\delta X_i^{(l)}(t')}}, \tag{58}$$

the closure equations are written as

$$\begin{aligned}2\nu(k^{(l)})^2V^{(l)}(0) = {} & 4\mathcal{C}(k^{(l)})^2V^{(l)}(0)(V^{(l+1)}(0)-V^{(l)}(0))\int_0^\infty \mathrm{d}\tau\, G^{(l+1)}(\tau)(G^{(l)}(\tau))^2 \\ & -4\mathcal{C}(k^{(l-1)})^2V^{(l-1)}(0)(V^{(l)}(0)-V^{(l-1)}(0))\int_0^\infty \mathrm{d}\tau\, G^{(l)}(\tau)(G^{(l+1)}(\tau))^2,\end{aligned} \tag{59}$$

$$\begin{aligned}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}+\nu(k^{(l)})^2\right\}G^{(l)}(\tau) = {} & -2\mathcal{C}(k^{(l)})^2(V^{(l+1)}(0)+V^{(l)}(0)) \\ & \times\int_0^\tau \mathrm{d}\tau'\, G^{(l)}(\tau')G^{(l)}(\tau-\tau')G^{(l+1)}(\tau') \\ & -2\mathcal{C}(k^{(l-1)})^2V^{(l-1)}(0)\int_0^\tau \mathrm{d}\tau'\, G^{(l)}(\tau-\tau')(G^{(l-1)}(\tau'))^2\end{aligned} \tag{60}$$

and

$$V^{(l)}(\tau) = V^{(l)}(0)G^{(l)}(\tau), \tag{61}$$

in place of (29)–(31) for the uniform system. Here, we have assumed statistical stationarity of the system, and $\overline{X_i^{(l)}(t)} = 0$.

Energy transfer function $T^{(l)}$ and energy flux function $\Pi^{(l)}$ are, respectively, defined as

$$T^{(l)}(t) = \left\{\frac{\mathrm{d}}{\mathrm{d}t}+\nu(k^{(l)})^2\right\}E^{(l)}(t), \tag{62}$$

$$\Pi^{(l)}(t) = \sum_{l'=l}^{l_{\max}} k^{(l')}T^{(l')}(t). \tag{63}$$

Noting that the energy spectrum is written by the one-time auto-correlation function as

$$E^{(l)} = \frac{N}{2\,k^{(l)}}V^{(l)}(0), \tag{64}$$

we obtain

$$\Pi^{(l)} = \frac{8\mathcal{C}}{N}(k^{(l-1)})^3E^{(l-1)}(k^{(l-1)}E^{(l-1)}-k^{(l)}E^{(l)})\int_0^\infty \mathrm{d}\tau(G^{(l-1)}(\tau))^2G^{(l)}(\tau) \tag{65}$$

and

$$\begin{aligned}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}+\nu(k^{(l)})^2\right\}G^{(l)}(\tau) = {} & -\frac{4\mathcal{C}}{N}(k^{(l)})^2(k^{(l+1)}E^{(l+1)}+k^{(l)}E^{(l)}) \\ & \times\int_0^\tau \mathrm{d}\tau' G^{(l)}(\tau')\,G^{(l)}(\tau-\tau')\,G^{(l+1)}(\tau') \\ & -\frac{4\mathcal{C}}{N}(k^{(l-1)})^3E^{(l-1)}\int_0^\tau \mathrm{d}\tau'\, G^{(l)}(\tau-\tau')(G^{(l-1)}(\tau'))^2\end{aligned} \tag{66}$$

from (59) and (60), respectively.

Now we solve the SDIP equations (65) and (66) in the inertial range, and show that the solution is consistent with the Kolmogorov similarity. First, we assume a scale similarity both in the energy spectrum and the response function as

$$E^{(l)} = A(k^{(l)})^{\alpha}, \qquad G^{(l)}(\tau) = g\left(\frac{\tau}{\tau^{(l)}}\right) \qquad \text{with } \tau^{(l)} = B(k^{(l)})^{\beta}. \tag{67}$$

In the inertial range, the viscous term in (66) is negligible, and the energy flux function is a constant equal to the energy dissipation rate, $\Pi^{(l)} = \epsilon$. Then, the SDIP equations (65) and (66), respectively, require the relations $4+2\alpha+\beta = 0$, $A^2B \propto \epsilon$ and $3+\alpha+2\beta = 0$, $AB^2 \propto \epsilon^0$, which lead to

$$\alpha = -\tfrac{5}{3}, \qquad \beta = -\tfrac{2}{3} \tag{68}$$

and

$$A = K\epsilon^{2/3}, \qquad B = \epsilon^{-1/3} \tag{69}$$

without loss of generality. Here, K is called the Kolmogorov constant of the present shell model (54). The exponents (68) represent the Kolmogorov similarity, i.e. the energy spectrum is proportional to $(k^{(l)})^{-5/3}$ and the correlation time $(k^{(l)})^{-2/3}$. Then, (65) and (66), respectively, yield

$$1 = \frac{8K^2\mathcal{C}}{N}(1 - D^{-2/3}) \int_0^{\infty} \mathrm{d}t\,(g(t))^2 g(D^{2/3}t) \tag{70}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = -\frac{4K\mathcal{C}}{N}\left\{(D^{-2/3}+1)\int_0^t \mathrm{d}t'\, g(t')g(t-t')g(D^{2/3}t') + D^{-4/3}\int_0^t \mathrm{d}t'\, g(t-t')(g(D^{-2/3}t'))^2\right\}. \tag{71}$$

The Kolmogorov constant K and the auto-correlation function $g(\tau/\tau^{(l)}) = V^{(l)}(\tau)/V^{(l)}(0)$ are estimated by solving (70) and (71). Then, we obtain $K = 1.01$, which is compared with the value 0.97 determined by the least mean square method of the energy spectrum in figure 10(*a*). The auto-correlation function is compared with the results of direct numerical simulation of (54) in figure 10(*b*). Predictions of both of K and g by SDIP agree with the simulation results in high precision. We may conclude, therefore, that SDIP applies to this shell model successfully. Incidentally the Kolmogorov constant K as well as the functional form of $g(t)$ depend on the model parameters D and $\mathcal{C}/N$.

Finally, we stress again that the variable $X_i^{(l)}$ of the shell model (54) may mimic the Lagrangian velocity rather than the Eulerian counterpart because the correlation time of the variable is proportional to $\epsilon^{1/3}(k^{(l)})^{-2/3}$ (see figure 10(*b*)) rather than $(k^{(l)})^{-1}$ as the Eulerian velocity [18]. In this sense, the present SDIP is of Lagrangian nature, and its success is consistent with that of the Lagrangian SDIP for the Navier–Stokes turbulence [9].

7. Concluding remarks

The second-order statistics of a quadratic nonlinear dynamical system are well described by SDIP. This approximation is based on the SDIP assumptions (see section 3.2) that the three-mode correlation is induced mainly by direct interactions between them, and that a removal of a single direct interaction may be treated as a perturbation from the true field. Sparseness of nonlinear couplings and largeness of the number of degrees of freedom play key roles in these assumptions. We applied them to the model equation (3), and obtained the SDIP equations for the two-mode correlation and the average response functions, which give excellent predictions of the auto-correlation function both in the statistically stationary and non-stationary states (figures 4 and 9), when the coupling density ρ (i.e. the number of direct interactions between two modes) is sufficiently smaller than the square root of the number N of degrees of freedom (figure 5). We emphasize that validity of SDIP is related neither with the strength of nonlinearity (the Reynolds number) nor with the Gaussianity of pdf of dynamical variable. Indeed, the Reynolds number in figure 4 is infinitely large, and three pdfs of X_i in the cases of figure 9 are far from Gaussian.

The SDIP is also applied successfully to a shell model of turbulence with sparse coupling. The solution to SDIP equations agrees excellently with the Kolmogorov similarity law of energy spectrum in the inertial range (figure 10). The dynamical variable in this model has Lagrangian nature, and this result is consistent with the fact that the Lagrangian SDIP [9] for the Navier–Stokes turbulence is successful but the Eulerian counterpart fails to express the inertial range properties. In the application of SDIP to a dynamical system, the choice of the representative is important in addition to the sparseness of nonlinear coupling. The Lagrangian velocity is the appropriate representative for the Navier–Stokes turbulence. We must admit, however, that it is not completely understood why the Lagrangian velocity should be employed instead of the Eulerian one, though this problem has been discussed extensively in terms of the Galilean invariance, the sweeping effect, etc.

Appendix A. An example of intermediate coupling density system

We give an example of a system with intermediate nonlinear coupling density using the Burgers equation,

$$\frac{\partial}{\partial t}v(x,t) = v(x,t)\frac{\partial}{\partial x}v(x,t) + \nu\frac{\partial^2}{\partial x^2}v(x,t), \tag{A1}$$

where $0 \leqslant x \leqslant L$. In the case of the periodic boundary condition, the nonlinear coupling density ρ is equal to 1 similarly to (1) because $v(x,t)$ can be expanded as $v(x,t) = \sum_k \tilde{v}(k,t)\,\mathrm{e}^{\mathrm{i}kx}$. However, when the boundaries are rigid,

$$v(0,t) = v(L,t) = 0, \tag{A2}$$

the field is expressed, in general, by

$$v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\tilde{v}_n(t)\sin\left(\frac{n\pi}{L}x\right). \tag{A3}$$

Then, (A1) is written in terms of $\tilde{v}_n$ as

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t} + \nu\left(\frac{n\pi}{L}\right)^2\right\}\tilde{v}_n(t) = \frac{1}{L}\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{q\pi}{L}\tilde{v}_p\tilde{v}_q(\delta_{p+q,n} + \delta_{p-q,n} - \delta_{q-p,n}). \tag{A4}$$

This system has intermediate coupling density, $\rho \approx 2$.

Appendix B. SDIP formulation for a system with non-zero mean

Similarly to (19), the response function $G_{in}(t,t')$ is also decomposed as

$$G_{in}(t,t') = G^{(0)}_{in/i_0j_0k_0}(t,t') + G^{(1)}_{in/i_0j_0k_0}(t,t'), \tag{B1}$$

where the two quantities on the right-hand side are, respectively, governed by

$$\frac{\partial}{\partial t}G^{(0)}_{in/i_0j_0k_0}(t,t') = \sum_{j=1}^{N}\sum_{k=1}^{N} 2\,C_{ijk}x_j(t)G^{(0)}_{kn/i_0j_0k_0}(t,t') - \nu G^{(0)}_{in/i_0j_0k_0}(t,t')$$
$$+ \sum_{\substack{j=1 \\ \{i,j,k\}\neq\{i_0,j_0,k_0\}}}^{N}\sum_{k=1}^{N} 2\,C_{ijk}Y_j(t)G^{(0)}_{kn/i_0j_0k_0}(t,t') \tag{B2}$$

and

$$\frac{\partial}{\partial t}G^{(1)}_{in/i_0j_0k_0}(t,t') = \sum_{j=1}^{N}\sum_{k=1}^{N} 2\,C_{ijk}x_j(t)G^{(1)}_{kn/i_0j_0k_0}(t,t') - \nu G^{(1)}_{in/i_0j_0k_0}(t,t')$$
$$+ \sum_{\substack{j=1 \\ \{i,j,k\}\neq\{i_0,j_0,k_0\}}}^{N}\sum_{k=1}^{N} 2\,C_{ijk}Y_j(t)G^{(1)}_{kn/i_0j_0k_0}(t,t')$$
$$+\delta_{ii_0}2\,C_{i_0j_0k_0}\{Y_{j_0}(t)G^{(0)}_{k_0n/i_0j_0k_0}(t,t') + Y_{k_0}(t)G^{(0)}_{j_0n/i_0j_0k_0}(t,t')\}$$
$$+\delta_{ij_0}2\,C_{j_0k_0i_0}\{Y_{k_0}(t)G^{(0)}_{i_0n/i_0j_0k_0}(t,t') + Y_{i_0}(t)G^{(0)}_{k_0n/i_0j_0k_0}(t,t')\}$$
$$+\delta_{ik_0}2\,C_{k_0i_0j_0}\{Y_{i_0}(t)G^{(0)}_{j_0n/i_0j_0k_0}(t,t') + Y_{j_0}(t)G^{(0)}_{i_0n/i_0j_0k_0}(t,t')\}, \tag{B3}$$

with $G^{(0)}_{in/i_0j_0k_0}(t,t) = \delta_{in}$ and $G^{(1)}_{in/i_0j_0k_0}(t,t) = 0$. Then, by employing $G^{(0)}_{in}(t,t')$ as Green's function, we can express formal solutions of (21) and (B3) as

$$Y^{(1)}_{i/i_0j_0k_0}(t|t_0) = \int_{t_0}^{t} \mathrm{d}t' \big\{2\,G^{(0)}_{ii_0/i_0j_0k_0}(t,t')C_{i_0j_0k_0}Y^{(0)}_{j_0/i_0j_0k_0}(t')Y^{(0)}_{k_0/i_0j_0k_0}(t')$$
$$+2\,G^{(0)}_{ij_0/i_0j_0k_0}(t,t')C_{j_0k_0i_0}Y^{(0)}_{k_0/i_0j_0k_0}(t')Y^{(0)}_{i_0/i_0j_0k_0}(t')$$
$$+2\,G^{(0)}_{ik_0/i_0j_0k_0}(t,t')C_{k_0i_0j_0}Y^{(0)}_{i_0/i_0j_0k_0}(t')Y^{(0)}_{j_0/i_0j_0k_0}(t')\big\}, \tag{B4}$$

and

$$G^{(1)}_{in}(t,t') =$$
$$\int_{t'}^{t} \mathrm{d}t'' \big[2\,G^{(0)}_{ii_0/i_0j_0k_0}(t,t'')C_{i_0j_0k_0}\big\{Y_{j_0}(t'')G^{(0)}_{k_0n/i_0j_0k_0}(t'',t') + Y_{k_0}(t'')G^{(0)}_{j_0n/i_0j_0k_0}(t'',t')\big\}$$
$$+2\,G^{(0)}_{ij_0/i_0j_0k_0}(t,t'')C_{j_0k_0i_0}\big\{Y_{k_0}(t'')G^{(0)}_{i_0n/i_0j_0k_0}(t'',t') + Y_{i_0}(t'')G^{(0)}_{k_0n/i_0j_0k_0}(t'',t')\big\}$$
$$+2\,G^{(0)}_{ik_0/i_0j_0k_0}(t,t'')C_{k_0i_0j_0}\big\{Y_{i_0}(t'')G^{(0)}_{j_0n/i_0j_0k_0}(t'',t') + Y_{j_0}(t'')G^{(0)}_{i_0n/i_0j_0k_0}(t'',t')\big\}\big]. \tag{B5}$$

Decompositions (19) and (B1) are naive extensions to a system with non-zero mean of that with zero mean $x_i = 0$ (see [10]). The resultant SDIP equations (24)–(26) are the same between the two cases as shown below.

The first term on the right-hand side of the evolution equation (14) for the auto-correlation function is written as

$$(\text{First term on the rhs of (14)}) = \sum_{j=1}^{N}\sum_{k=1}^{N} 2\,C_{ijk}x_j(t)\overline{Y^{(0)}_{k/ijk}(t)Y^{(0)}_{i/ijk}(t')}$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{N} 2\,C_{ijk}x_j(t)\overline{Y^{(1)}_{k/ijk}(t)Y^{(0)}_{i/ijk}(t')}$$
$$+\sum_{j=1}^{N}\sum_{k=1}^{N} 2\,C_{ijk}x_j(t)\overline{Y^{(0)}_{k/ijk}(t)Y^{(1)}_{i/ijk}(t')}, \tag{B6}$$

where higher-order terms have been neglected. The first term of (B6) vanishes because $Y^{(0)}_{i/ijk}$ and $Y^{(0)}_{k/ijk}$ are statistically independent of each other, and because $\overline{Y^{(0)}_{i/ijk}(t)} = 0$. Furthermore, by substituting the formal solution (B4) of $Y^{(1)}_{i/i_0j_0k_0}(t)$, the other two terms are also shown to be zero. For example, the second term of (B6) is rewritten as

(Second term of (B6))

$$= \sum_{j=1}^{N}\sum_{k=1}^{N} 4\, C_{ijk} x_j(t) \int_{t_0}^{t} \mathrm{d}t'' \Big\{ C_{ijk} \overline{G^{(0)}_{ki/ijk}(t,t'') Y^{(0)}_{j/ijk}(t'') Y^{(0)}_{k/ijk}(t'') Y^{(0)}_{i/ijk}(t')}$$
$$+ C_{jki} \overline{G^{(0)}_{kj/ijk}(t,t'') Y^{(0)}_{k/ijk}(t'') Y^{(0)}_{i/ijk}(t'') Y^{(0)}_{i/ijk}(t')}$$
$$+ C_{kij} \overline{G^{(0)}_{kk/ijk}(t,t'') Y^{(0)}_{i/ijk}(t'') Y^{(0)}_{j/ijk}(t'') Y^{(0)}_{i/ijk}(t')} \Big\}, \tag{B7}$$

where we have assumed that $G^{(0)}_{in/ijk}$ and $Y^{(0)}_{j/ijk}$ are statistically independent of each other because they have no direct interaction. (Validity of this assumption has been checked numerically [10].) Then, all the terms in (B7) vanish because of independency between $X^{(0)}_{i/ijk}$, $X^{(0)}_{j/ijk}$ and $X^{(0)}_{k/ijk}$. Thus, it is shown that

$$(\text{First term on the rhs of } (14)) = 0. \tag{B8}$$

Next, the second term on the right-hand side of (14) is written as

$$(\text{Second term on the rhs of } (14)) = \sum_{j=1}^{N}\sum_{k=1}^{N} 2\, C_{ijk} \overline{Y^{(0)}_{j/ijk}(t) Y^{(0)}_{k/ijk}(t) Y^{(0)}_{i/ijk}(t')}$$
$$+ \sum_{j=1}^{N}\sum_{k=1}^{N} 4\, C_{ijk} \overline{Y^{(1)}_{j/ijk}(t) Y^{(0)}_{k/ijk}(t) Y^{(0)}_{i/ijk}(t')}$$
$$+ \sum_{j=1}^{N}\sum_{k=1}^{N} 2\, C_{ijk} \overline{Y^{(0)}_{j/ijk}(t) Y^{(0)}_{k/ijk}(t) Y^{(1)}_{i/ijk}(t')}. \tag{B9}$$

where higher-order terms have been neglected. The first term vanishes. A straightforward calculation after substitution of the formal solution (B4) into the second and the third terms leads to the right-hand side of (24). The SDIP equation (25) for one-time correlation function is derived from (15) by the same procedures.

The SDIP formulation for the response function equation (18) is also straightforward. Since only auto-response functions $G(t,t') = \overline{G_{ii}(t,t')}$ appear in the SDIP equations (24) and (25), we put $i = n$ in (18). Then, the first term on the right-hand side of (18) is written as

$$(\text{First term on the rhs of } (18)) = \sum_{j=1}^{N}\sum_{k=1}^{N} 2\, C_{ijk} x_j(t) \Big\{ \overline{G^{(0)}_{ki/ijk}(t,t')} + \overline{G^{(1)}_{ki/ijk}(t,t')} \Big\}. \tag{B10}$$

It can be shown, by substituting (B5) into the second term of (B10), that both terms of (B10) vanish because $\overline{G^{(0)}_{ki/ijk}(t,t')} = 0$ and $\overline{Y_i(t)} = 0$. Next, the second term on the right-hand side of (18) is rewritten by substituting the direct-interaction decompositions (B1) as

$$(\text{Second term on the rhs of } (18)) = \sum_{j=1}^{N}\sum_{k=1}^{N} 2\, C_{ijk} \Big\{ \overline{Y_j(t) G^{(0)}_{ki/ijk}(t,t')} + \overline{Y_j(t) G^{(1)}_{ki/ijk}(t,t')} \Big\}. \tag{B11}$$

The first term of (B11) vanishes, since Y_j and $G^{(0)}_{ki/ijk}$ are statistically independent of each other, and since $\overline{Y_j(t)} = 0$. Using (B5), the second term of (B11) is shown to lead to the right-hand side of (26). Thus, we arrive at (26).

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