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A multiplicative process of material line stretching by turbulence†

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Abstract. Statistics of material lines passively advected in stationary homogeneous isotropic turbulence of an incompressible fluid are investigated from the viewpoint that material line stretching is a multiplicative process. It is shown by a simple argument based upon this perspective that material line elements cannot be treated equivalently in the line statistics. In other words, a frequently used assumption (Batchelor G K 1952 *Proc. R. Soc.* A **213** 349–66) of their equivalence is not valid in general. The invalidity of this assumption is further confirmed by numerically estimating the correct stretching rate of material lines as 0.17/(Kolmogorov time) which would be 0.13/(Kolmogorov time) if the Batchelor assumption were valid. This substantial difference is a manifestation of the fact that turbulence has spatial inhomogeneity of order of the Kolmogorov length and none-zero finite correlation time of order of the Kolmogorov time, reflecting the existence of small-scale coherent structures.

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Contents

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1. Introduction—the Batchelor assumption

An object (line, surface or volume) which always consists of the same fluid particles is called a material object. Material objects are deformed in a complicated way by a turbulent velocity field as shown in figures 1 and 2, which are typical results of direct numerical simulation of a material line (see section 4) in stationary homogeneous turbulence. It was Batchelor who first challenged this complex system by introducing several important conjectures and assumptions. The most important assumption, which is called the Batchelor assumption in this paper, made in his pioneer work [1] is stated as:

> *In view of the homogeneity of turbulence the statistical behaviour of all elements of the above line is the same, ... Consequently it will be sufficient to confine our discussion to the statistical behaviour of the material line element ...,*

which reads that a material line (or surface) in stationary homogeneous turbulence is composed of a number of statistically equivalent elements, and therefore the mean value of a quantity over the line (or surface) might be simply calculated by the arithmetic mean over the elements with equal weight. The Batchelor assumption has been used frequently in studies of the statistics of material objects in turbulence because it makes both the theoretical and numerical analysis extremely simple. Especially, it has been quite common that a set of infinitesimal material elements is used for consideration of statistics of material objects by direct numerical simulations. Batchelor conjectured also [1] that the length of a material line or the area of a material surface increases exponentially, except in a very early stage of the evolution. The exponential form of stretching has been confirmed numerically [2, 3], but it has been shown [4] that the average of the stretching rate of material lines estimated based on the Batchelor assumption underestimates the correct value by about 24%. This implies that the Batchelor assumption is not valid in general. In other words, material elements can never be statistically equivalent even in homogeneous turbulence.

The main purpose of this paper is to clarify the limitation of the Batchelor assumption by regarding the stretching of material objects as a multiplicative process. The underestimation of the stretching rate, which gives evidence of the invalidity of the assumption, is understood from the property of multiplicative processes that correlation between a single step and the product of all steps remains non-zero finite even when the number of steps increases infinitely. This property may be called the non-decaying correlation of a multiplicative process. It will be shown in what follows that if the correlation time of stretching were zero or if the velocity gradient field had no spatial fluctuation then the Batchelor assumption would be valid. In real turbulence, however, the stretching of material objects has spatial nonuniformity and non-zero finite correlation time, which are the Kolmogorov length and timescale, respectively.

This paper is organized as follows. After describing the governing equations of this system in the next section (section 2), the invalidity of the Batchelor assumption is mathematically examined in section 3 from the viewpoint that the material line stretching is a multiplicative

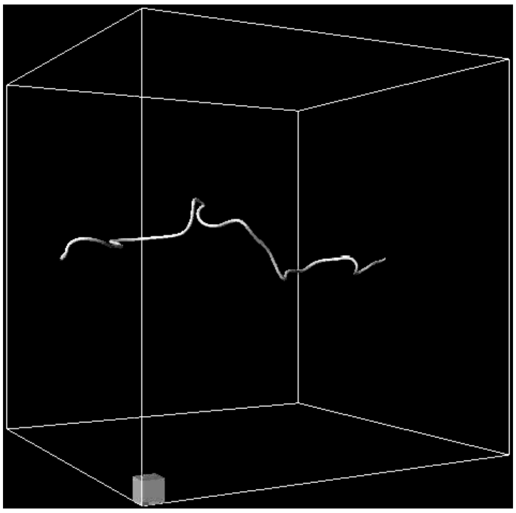

Figure 1. Temporal evolution between $t = 0$ and 4 of a material line in turbulence at $R_\lambda = 56$ ($\tau_\eta = 2.0 \times 10^{-1}$). The entire box is the periodic box of the velocity field, and the side of the small box indicates ten times the Kolmogorov length.

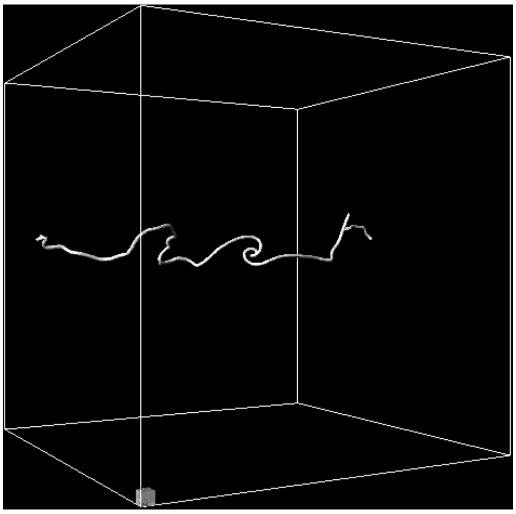

Figure 2. Temporal evolution between $t = 0$ and 4 of a material line in turbulence at $R_\lambda = 84$ ($\tau_\eta = 1.4 \times 10^{-1}$). The entire box is the periodic box of the velocity field, and the side of the small box indicates ten times the Kolmogorov length.

process, and checked by the use of direct numerical simulations in section 4. Concluding remarks are given in section 5, where another assumption made by Batchelor on the probability density function (pdf) of the length of material line elements is briefly examined.

2. Governing equations

The temporal evolution of a material object (line, surface or volume) is described by the advection equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{x}_m(t) = \boldsymbol{u}(\boldsymbol{x}_m(t), t), \tag{1}$$

where $\boldsymbol{x}_m(t)$ is a position vector on the material object and $\boldsymbol{u}(\boldsymbol{x},t)$ is the velocity field of an incompressible fluid. Hereafter, we assume that the velocity field is governed by the equation of continuity,

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x},t) = 0 \tag{2}$$

and the Navier–Stokes equation,

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}(\boldsymbol{x},t) \cdot \nabla \right) \boldsymbol{u}(\boldsymbol{x},t) = -\frac{1}{\rho} \nabla p(\boldsymbol{x},t) + \nu \nabla^2 \boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{f}(\boldsymbol{x},t), \tag{3}$$

where ρ is the constant density of the fluid, ν is the kinematic viscosity, $p(\boldsymbol{x},t)$ is the pressure and $\boldsymbol{f}(\boldsymbol{x},t)$ is an external force. This external force makes turbulence statistically stationary. We impose the periodic boundary conditions for the velocity field in three orthogonal directions, therefore turbulence can be statistically homogeneous and isotropic. It has been assumed that the material objects are passive, that is, there is no reaction from the material objects to the fluid motion, and the effect of molecular diffusivity is also neglected. These assumptions are introduced for simplicity, but the system may be meaningful when the molecular diffusivity is sufficiently small compared with the turbulence mixing. Hence, passive material objects have been extensively studied by many authors for the past 50 years as one of the most fundamental topics in turbulence mixing research, and are quite important in applications to such flows accompanied with chemical reaction or combustion, diffusion of pollutant in the environment, and so on.

3. Statistics on the material line

3.1. The line average and line-element average

In the following, we restrict ourselves to the deformation of material lines in turbulence because the problem appearing in the deformation of material surfaces is essentially the same. The characteristic length scale of deformation of material lines may be of order of the Kolmogorov length $\eta = \nu^{\frac{3}{4}} \epsilon^{-\frac{1}{4}}$, where ϵ is the energy dissipation rate per unit mass. Therefore, a material line is approximated by a chain of material line elements which are much shorter than η. Let $\Delta^{(i)}(t) \ll \eta$ be the length of the ith line element ($i = 1, 2, \ldots, I$) at time t. Then the total length of the material line may be expressed as

$$L(t) = \int_L \mathrm{d}\ell = \sum_{i=1}^{I} \Delta^{(i)}(t), \tag{4}$$

where $\int_L \mathrm{d}\ell$ denotes the line integral along a material line. Then, the line average of a quantity g along the material line,

$$\langle g \rangle_{\mathrm{line}} = \frac{1}{L} \int_L g \, \mathrm{d}\ell, \tag{5}$$

may be approximated by

$$\langle g \rangle_{\mathrm{line}} = \frac{\sum_{i=1}^{I} g^{(i)}(t)\, \Delta^{(i)}(t)}{\sum_{i=1}^{I} \Delta^{(i)}(t)}. \tag{6}$$

Here, $g^{(i)}$ is the value of g on the ith line element. The Batchelor assumption of the statistical equivalence of all the elements immediately leads to the equivalence of this line average and the line-element average, the latter of which is defined by

$$\langle g \rangle_{\text{line-element}} = \frac{1}{I} \sum_{i=1}^{I} g^{(i)}(t). \tag{7}$$

The line average (6) is taken along a material line with probability weight proportional to the element length $\Delta^{(i)}$, while the line-element average (7) is taken over the line elements with equal probability weight.

The line average is written, in terms of the line-element average, as

$$\langle g\rangle_{\text{line}} = \frac{\langle g\Delta\rangle_{\text{line-element}}}{\langle\Delta\rangle_{\text{line-element}}}, \tag{8}$$

which shows that the Batchelor assumption requires the statistical independency between the quantity $g^{(i)}$ and the element length $\Delta^{(i)}$. Note that the line average and the line-element average of $g^{(i)}$ give, in general, different values if $g^{(i)}$ and $\Delta^{(i)}$ are correlated.

3.2. An example: stretching rate

As an example of $g^{(i)}$, we take the stretching rate,

$$\gamma_e^{(i)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\log\Delta^{(i)}(t) \tag{9}$$

of each line element. The line average of $\gamma_e^{(i)}$ gives the stretching rate,

$$\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\,\log L(t) \tag{10}$$

of the material line itself, where L is the total length expressed as (4). It follows from (4), (6), (9) and (10) that

$$\gamma(t) = \langle\gamma_e(t)\rangle_{\text{line}}. \tag{11}$$

It should be mentioned, in passing, that the stretching rate γ of the material line has been the target of many theories [1, 5] and direct numerical simulations [2]–[4] as one of the most important quantities which characterize the strength of turbulence mixing.

The integral of (9),

$$\Delta^{(i)}(t) = \Delta^{(i)}(0)\,\exp\left[\int_0^t \gamma_e^{(i)}(t')\,\mathrm{d}t'\right], \tag{12}$$

suggests that $\Delta^{(i)}(t)$ and $\gamma^{(i)}(t)$ may be positively correlated, which will be confirmed theoretically in the next section and numerically in section 4.5. If so, the line-element average of $\gamma_e^{(i)}(t)$ should be smaller than the line average, i.e.

$$\langle\gamma_e\rangle_{\text{line-element}} < \langle\gamma_e\rangle_{\text{line}} = \gamma. \tag{13}$$

This is contradictory to the result obtained from the Batchelor assumption.

3.3. Multiplicative process, non-decaying correlation and the difference between the two averages

The mathematical basis of positive correlation between the stretching rate $\gamma_e^{(i)}(t)$ and the element length $\Delta^{(i)}(t)$ is explained in this section by regarding the stretching of material lines as a multiplicative process. First of all, we assume that this process has non-zero finite correlation time, τ_c, which will be defined by (21) below, in other words, the stretching process is not white in time. This correlation time is expected to be of order of the Kolmogorov time as will be confirmed numerically in section 4.4. Then the time integration in (12) may be divided as

$$\begin{aligned}\Delta^{(i)}(t) &= \Delta^{(i)}(0)\,\exp\left[\int_0^{\tau_c}\gamma_e^{(i)}(t')\,\mathrm{d}t'\right]\exp\left[\int_{\tau_c}^{2\tau_c}\gamma_e^{(i)}(t')\,\mathrm{d}t'\right]\cdots\exp\left[\int_{t-\tau_c}^{t}\gamma_e^{(i)}(t')\,\mathrm{d}t'\right]\\ &= \Delta^{(i)}(0)\,\sigma^{(i)}(0,\tau_c)\,\sigma^{(i)}(\tau_c,2\tau_c)\,\cdots\,\sigma^{(i)}(t-\tau_c,t),\end{aligned} \tag{14}$$

where $\sigma^{(i)}(t_1, t_2)$ is defined by

$$\sigma^{(i)}(t_1, t_2) = \exp\left[\int_{t_1}^{t_2} \gamma_e^{(i)}(t')\,\mathrm{d}t'\right] = \frac{\Delta^{(i)}(t_2)}{\Delta^{(i)}(t_1)} \tag{15}$$

and is called the stretched factor. A pair of stretched factors separated by a period longer than τ_c are statistically independent of each other. Therefore, (14) can be regarded as a product of mutually independent random variables with a common probability distribution. In this sense, the stretching of a material line is a multiplicative process. An important feature of multiplicative processes is that the correlation between one variable and the product of all variables never decay to zero even in the limit of a large number of steps. This may be called the non-decaying correlation in a multiplicative process.

As seen in the preceding section, the finite correlation between $\Delta(t)$, which is the result of the whole multiplicative process, and $\gamma_e(t)$, which is attached only to the latest step of the process, is the cause of the difference between the line average and the line-element average of γ_e. Hence, it is the property of the non-decaying correlation that yields the difference. Indeed, by substituting (14) into the line average (8) of the stretching rate ($g = \gamma_e$), we obtain

$$\langle\gamma_e\rangle_{\text{line}} = \frac{\langle\gamma_e\Delta\rangle_{\text{line-element}}}{\langle\Delta\rangle_{\text{line-element}}} = \frac{\langle\gamma_e(t)\,\sigma(t-\tau_c, t)\rangle_{\text{line-element}}}{\langle\sigma(t-\tau_c, t)\rangle_{\text{line-element}}} \tag{16}$$

because of the statistical independency between the stretched factors separated more than τ_c. Equation (16) tells us that the correlation between the stretched factor $\sigma(t-\tau_c, t)$ at the latest period and $\gamma_e(t)$ contributes to that between $\Delta(t)$ and $\gamma_e(t)$, and that this correlation does not decay in time. Hence, the difference between the two averages also never vanish even after a sufficiently long period.

It is seen from (16) that if the stretched factor $\sigma(t-\tau_c, t)$, which is the probability weight in the line average, is uniform for all line elements, then the line average and the line-element average are identical. Such a uniformity of the stretched factor is possible in the following two cases. One is when the velocity gradient field is uniform over the whole domain. In this trivial case, although unlikely in actual turbulence, the stretched factor σ, as well as the stretching rate γ_e, are the same for all line elements. The other case is when the stretching process is white in time, i.e. $\tau_c = 0$. In this case, the stretched factor is uniform and equal to unity because $\sigma(t, t) = 1$.

A more quantitative estimation of the difference between the two averages is given here. If the stretched factor in (16) is approximated by $\sigma \sim \mathrm{e}^{\gamma_e\tau_c}$, (16) may be written as

$$\langle\gamma_e\rangle_{\text{line}} \sim \frac{\langle\gamma_e\,\mathrm{e}^{\gamma_e\tau_c}\rangle_{\text{line-element}}}{\langle\mathrm{e}^{\gamma_e\tau_c}\rangle_{\text{line-element}}}, \tag{17}$$

which leads, by the Taylor expansion of $\mathrm{e}^{\gamma_e\tau_c}$, to

$$\langle\gamma_e\rangle_{\text{line}} - \langle\gamma_e\rangle_{\text{line-element}} = \mathrm{O}((\langle\gamma_e^2\rangle_{\text{line-element}} - \langle\gamma_e\rangle_{\text{line-element}}^2)\,\tau_c). \tag{18}$$

This implies that the amount of underestimation by the line-element average of the stretching rate is proportional to both the variance and the auto-correlation time of the stretching rate. Recall that the difference stems from the nonuniformity of the stretched factor, which is the probability weight in the line average. It is interesting to observe that the nonuniformity may be estimated by the product of the instantaneous fluctuation of the stretching rate and its auto-correlation time. It will be seen in section 4.4 that real turbulence has a non-zero finite variance of the stretching rate of $\mathrm{O}(0.01\,{\tau_\eta}^{-2})$ and its non-zero finite auto-correlation time of $\mathrm{O}(\tau_\eta)$, where $\tau_\eta = \epsilon^{-\frac{1}{2}}\,\nu^{\frac{1}{2}}$ is the Kolmogorov time.

4. Numerical check

In this section, the argument on the stretching of material lines in the preceding section is confirmed by the use of direct numerical simulations. After describing the numerical method of simulations in section 4.1 and showing typical results of temporal evolution of material lines in section 4.2, we numerically check the substantial difference between the line and the line-element averages of the stretching rate in section 4.3, the finiteness of the correlation time of stretching process in section 4.4, and the non-decaying positive correlation of this multiplicative process in section 4.5.

4.1. Numerical method

The system of governing equations (1)–(3) of the material lines and velocity field is simultaneously integrated by the use of a fourth-order Runge–Kutta scheme for time derivatives and the Fourier spectral method for spatial derivatives. The time increment of integration for (1) is the twice that for (3) and the aliasing interactions caused by the nonlinear term in (3) are removed by the phase shift method. The amplitudes of the Fourier components of velocity at wavenumbers smaller than $k_f(= 2.5)$ are kept constant, while their phases evolve temporally according to the governing equation. This replaces an effective forcing $\boldsymbol{f}(\boldsymbol{x}, t)$ in large scales. The right-hand side of (1) is estimated by the 4^3-point Lagrangian interpolation of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ at the numerical grid points.

We have carried out two kinds of numerical simulation: the line simulation and the line-element simulation. In the former, a material line consists of I elements, i.e. $I + 1$ nodes, and the position $\boldsymbol{x}^{(i)}(t)$ $(i = 0, 1, \ldots, I)$ of each node is governed by (1). To keep the numerical accuracy against the line stretching, the distance between a pair of successive nodes is always kept sufficiently smaller than the Kolmogorov length by inserting a new node at the centre of a line element as soon as it becomes longer than a threshold (1.5 times the numerical grid width). In the line-element simulation, we track many material line elements, each of which consists of pairs of two advected points with sufficiently close distance. In the same way as the line simulation, the position of a point is advected according to (1). At every numerical timestep, the length of each line element is renormalized to be kept as short as the numerical grid width.

We present here numerical results of two different sets of parameters: the kinematic viscosities are 5×10^{-3} and 2.5×10^{-3}, the Taylor-micro-scale Reynolds numbers $R_\lambda = \sqrt{20/3\nu\,\epsilon}\,\mathcal{E}$, where $\mathcal{E}$ is the turbulent energy per unit mass, are 56 and 84 on the temporal average, the Kolmogorov times are 2.0×10^{-1} and 1.4×10^{-1}, the Kolmogorov lengths are 3.2×10^{-2} and 1.9×10^{-2} on the temporal average, respectively. These Reynolds numbers are small enough that the smallest-scale turbulent motions, which play crucial roles in the stretching of material lines, are well resolved in both cases. The resolution is $N^3 = 128^3$, and the period of the velocity field is 2π in all three directions.

4.2. Exponential stretching of material lines

Typical results of line simulations at the two Reynolds numbers are shown in figures 1 and 2, where temporal evolutions of an initially straight material line are drawn in the period from $t = 0$ to 4. It is observed in these movies that the material line is strongly deformed by turbulence, and that the deformation is more rapid in the higher Reynolds number flow. This is reasonable because the lines are deformed strongly by the Kolmogorov-scale eddies with the Kolmogorov timescale. This timescale is shorter in a higher Reynolds number flow if the eddy turnover time

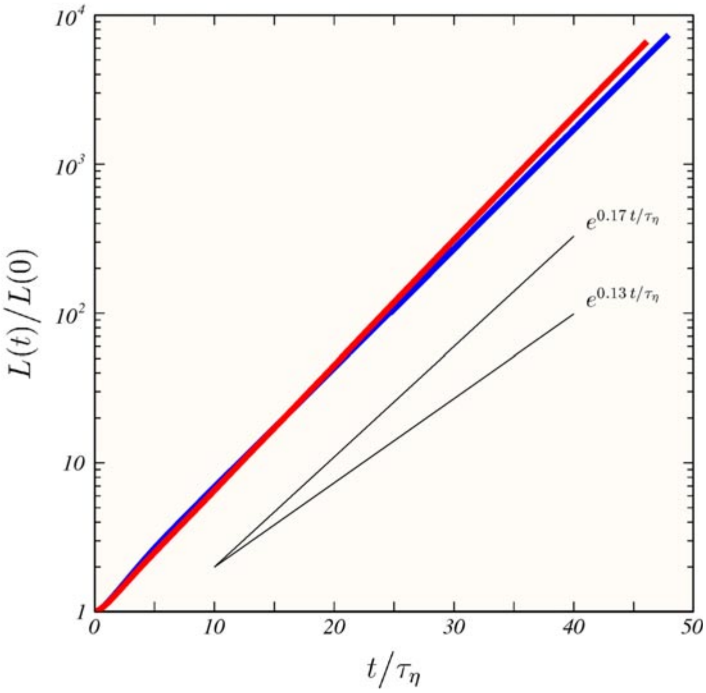

Figure 3. Total length of a material line. Blue, $R_\lambda = 56$; red, $R_\lambda = 84$.

is common as in the present cases. In order to confirm this Kolmogorov similarity with respect to the line stretching, we plot in figure 3 the ensemble average of the total line length over ten different realizations of turbulence, where the time is normalized by the Kolmogorov time. It is observed that the blue line ($R_\lambda = 56$) and the red one ($R_\lambda = 84$) overlap with each other, and that the total length of material lines is stretched exponentially except in the initial stage. The mean of stretching rate is around $\gamma = 0.17{\tau_\eta}^{-1}$ irrespective of the Reynolds number.

4.3. The difference between line and line-element averages

Let us move to numerical checks of the mathematical discussion given in section 3. First, we plot temporal evolutions of line average (solid curves) and line-element average (dashed curves) of the stretching rate $\gamma_e^{(i)}(t)$ for two different Reynolds numbers $R_\lambda = 56$ (red curves) and 84 (blue curves) in figure 4(a). The line-element average is estimated by the line-element simulations of 20 different initial conditions of turbulence and N^3 line elements, while the line average by the line simulations of 20 realizations and N^2 material lines of almost constant (by chopping the end of lines at every time step) length of order of the integral scale of turbulence. The stretching rate is larger for the higher Reynolds number, and the line average is larger than the line-element average in each Reynolds number turbulence, as expected from (13). In figure 4(b), the time is normalized by the Kolmogorov time, and the stretching rate by its reciprocal. Both curves in the line average and line-element average are independent of the Reynolds number in this normalization. After the initial period of order of the Kolmogorov time (see the next section) the mean values are almost constant around $0.17{\tau_\eta}^{-1}$ in the line average, and $0.13{\tau_\eta}^{-1}$ in the line-element average. The former is the true value of the stretching rate of the material line as seen in figure 3, whereas the latter corresponds to the value 0.13–$0.14{\tau_\eta}^{-1}$ [2, 3] estimated by the traditional method with a set of infinitesimal line elements based upon the Batchelor assumption.

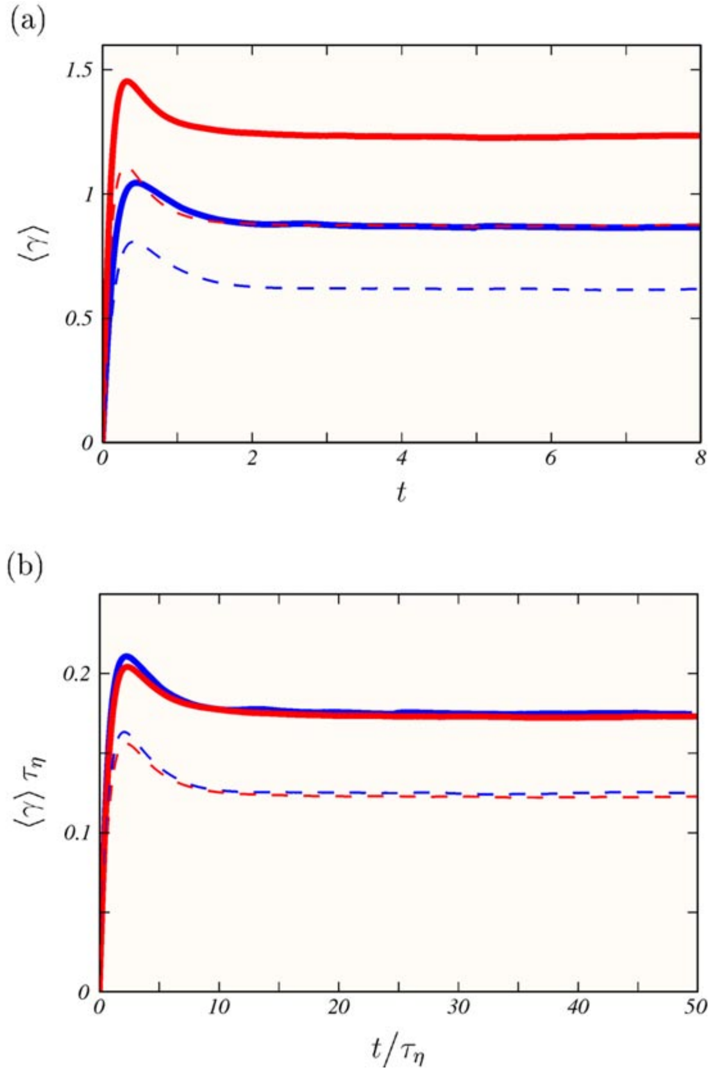

Figure 4. (a) Stretching rate of the material line. Solid curves, line average; dashed curves, line-element average. Blue, $R_\lambda = 56$; red, $R_\lambda = 84$. (b) Same as (a) but the time is normalized by the Kolmogorov time and the stretching rate by its reciprocal.

4.4. Finite correlation time

We have seen in the preceding section that the line-element average of the stretching rate underestimates the true value, i.e. the line average. Recall that the amount of this underestimation is estimated by (18) in section 3.3. As discussed there, the two averages would be identical if the stretching process had zero correlation time. In order to check the non-zero finite correlation time of the stretching process, the auto-correlation function of $\gamma_e^{(i)}(t)$ defined by

$$c_1(\tau) = \langle\, \tilde{\gamma}_e^{(i)}(t)\; \tilde{\gamma}_e^{(i)}(t+\tau)\, \rangle_{\text{line-element}} \tag{19}$$

with

$$\tilde{\gamma}_e^{(i)}(t) = \frac{\gamma_e^{(i)}(t) - \langle\gamma_e\rangle_{\text{line-element}}}{\left(\langle{\gamma_e}^2\rangle_{\text{line-element}} - \langle\gamma_e\rangle^2_{\text{line-element}}\right)^{1/2}} \tag{20}$$

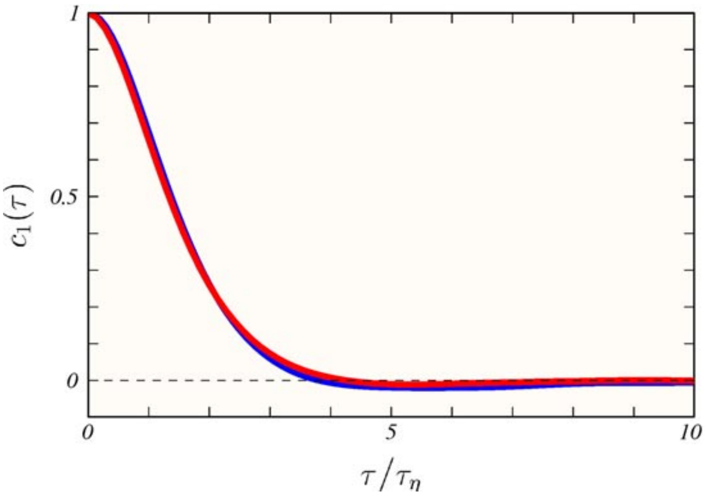

Figure 5. Auto-correlation function of the stretching rate of the material line element. Blue curve, $R_\lambda = 56$; red curve, $R_\lambda = 84$.

is plotted in figure 5. It is seen that the auto-correlation time, which hardly depends on the Reynolds number, is non-zero finite of order of the Kolmogorov time. Here, we define the auto-correlation time by

$$\tau_c = \frac{\displaystyle\int_0^\infty \tau c_1(\tau)\,\mathrm{d}\tau}{\displaystyle\int_0^\infty c_1(\tau)\,\mathrm{d}\tau}, \tag{21}$$

which amounts to $0.8\tau_\eta$. Incidentally, the variance of the stretching rate $\gamma_e^{(i)}$ is about $0.03\,{\tau_\eta}^{-2}$. Hence, the error estimation by (18) coincides roughly with the actual difference $0.04{\tau_\eta}^{-1}$ between the two averages as seen in figure 4(*b*). It should be stressed again that the non-zero finite correlation time $\tau_c = \mathrm{O}(\tau_\eta)$ of the stretching of material lines yields the finite difference between the line and line-element averages.

4.5. Non-decaying correlation between the stretching rate and the stretched factor

In this section, we confirm directly the non-decaying correlation between the stretching rate and the stretched factor. Note that the latter quantity depends on the entire stretching history, while the former is an instantaneous quantity at an observing time. This property of the non-decaying correlation between the entire history of a process and an instantaneous information of the process is one of the most prominent characteristics of multiplicative processes in contrast with additive processes. We plot, in figure 6, the normalized correlation coefficient between the stretched factor and the stretching rate defined by

$$c_2(t) = \langle\, \tilde{\sigma}^{(i)}(t)\, \tilde{\gamma}_e^{(i)}(t)\, \rangle_{\text{line-element}} \tag{22}$$

with (20) and

$$\tilde{\sigma}^{(i)}(t) = \frac{\hat{\sigma}^{(i)}(t) - \langle\hat{\sigma}\rangle_{\text{line-element}}}{(\langle\hat{\sigma}^2\rangle_{\text{line-element}} - \langle\hat{\sigma}\rangle^2_{\text{line-element}})^{1/2}}, \qquad \hat{\sigma}^{(i)}(t) = \frac{\sigma^{(i)}(t,0)}{\langle\sigma(t,0)\rangle_{\text{line-element}}}. \tag{23}$$

It is seen that the correlation stays almost constant after a sufficiently long time compared with the auto-correlation time, τ_c, of the stretching rate. This implies that all the elements never

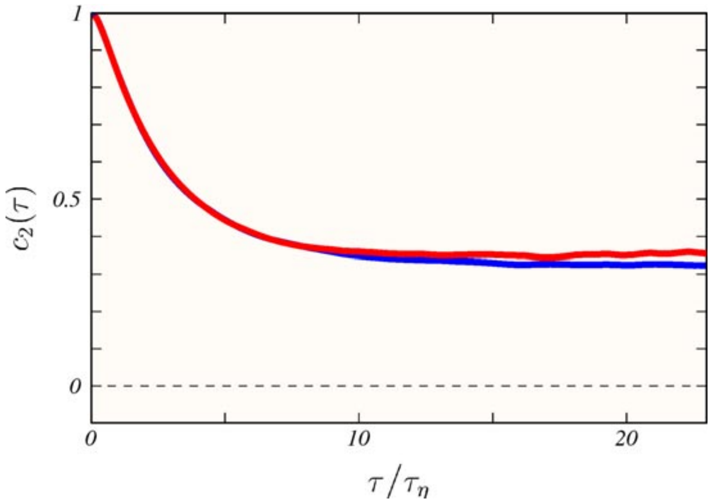

Figure 6. Correlation function between the stretching rate and the stretched factor. Blue curve, $R_\lambda = 56$; red curve, $R_\lambda = 84$.

forget their own history even after the correlation time period, and therefore they cannot be treated equivalently in contrast with the Batchelor assumption. It should be stressed again that the non-decaying correlation in a multiplicative process is quite important because this kind of probability process appears in many other systems, e.g., the energy cascade in the inertial subrange of turbulence.

5. Concluding remarks

Stretching of material line elements in turbulence is a multiplicative process in the sense that the ratio of lengths $\Delta(t)$ of a material line element at initial and observing times may be expressed as (14), i.e.

$$\frac{\Delta(t)}{\Delta(0)} = \frac{\Delta(t)}{\Delta(t-\tau_c)} \frac{\Delta(t-\tau_c)}{\Delta(t-2\tau_c)} \cdots \frac{\Delta(\tau_c)}{\Delta(0)}, \tag{24}$$

where τ_c is the auto-correlation time of this stretching process. Equation (24) implies that the ratio of two lengths, $\Delta(0)$ and $\Delta(t)$, is a product of many mutually independent random variables with a common probability distribution. One of the most important properties of multiplicative processes is the non-decaying correlation between a single step of the process and the product of many steps, i.e. the correlation between $\Delta(t)/\Delta(0)$ and $\Delta(t)/\Delta(t-\tau_c)$. Because of this property, the line elements cannot be statistically equivalent even after a much longer time than τ_c. Hence, in contrast to the Batchelor assumption (see section 1 which is the most important assumption in [1] and has been a foundation of many theoretical or numerical studies of material objects in turbulence) the nonuniform probability weight proportional to the length of each line element must be taken into account in the statistical treatment of a material line which consists of many line elements. It is also important that the difference between the two averages, with or without the probability weight, of a quantity accompanied with material lines is proportional to the correlation time τ_c. This implies that if turbulent velocity (gradient) field were white in time, the difference would disappear. In actual turbulence, of course, the correlation time is non-zero finite of order of the Kolmogorov time (figure 5).

As discussed throughout this paper, the Batchelor assumption has the essential limitation, and we cannot employ it for the statistics of any quantity which is accompanied with material objects and is correlated with the stretched factor. In [1] Batchelor introduced several important assumptions in addition to the one considered above. Here, we refer briefly to another assumption, i.e. the similarity of the pdf of a line element length $\varDelta(t)$, or equivalently the stretched factor $\sigma(t,0) = \varDelta(t)/\varDelta(0)$. The pdf $P(\sigma;t)$ of the stretched factor was assumed to be expressed, by a function Q independent of time, as

$$P(\sigma;t) = \frac{1}{\langle\sigma(t,0)\rangle}\, Q\left(\frac{\sigma(t,0)}{\langle\sigma(t,0)\rangle}\right), \tag{25}$$

but Girimaji and Pope [2] showed numerically that the similarity did not hold. This may be also understood by recalling that the stretching of material lines is a multiplicative process. It is easy to show that if the pdf has such a similarity, any-order moment of the element length is expressed, if it exists, by its mean as

$$\langle\sigma^n\rangle = \langle\sigma\rangle^n \qquad \text{(erroneous)} \tag{26}$$

because of a character of multiplicative processes,

$$\langle\sigma(m\tau_c,0)^n\rangle = \langle\sigma(\tau_c,0)^n\rangle^m. \tag{27}$$

Equation (26) is valid only when the pdf is the delta function, i.e. the stretched factor $\sigma(\tau_c,0)$ is uniform. This is possible if the velocity gradient field is uniform over the whole domain at all times, or if the stretching process is white in time ($\tau_c \to 0$). This condition of uniformity of the stretched factor is the same as the validity condition of the Batchelor assumption (section 1). However, real turbulence has the spatial fluctuation with finite correlation time of order of the Kolmogorov time.

Before closing this paper, we consider the physical meanings of the invalidity of Batchelor's assumptions, which is caused by the nonuniformity of the stretched factor, $\sigma(t-\tau_c,t)$. This nonuniformity stems from the finite spatial fluctuation of velocity gradient field with non-zero finite correlation time. It should be noted that such a fluctuation implies the existence of the small-scale coherent structures of the velocity gradient field, i.e. the vorticity and strain. The statistics of material object deformations are, therefore, concerned with the small-scale coherent structures. The limitation of the Batchelor assumption stimulates us to simulate numerically finite size material objects instead of the infinitesimal material elements, and the visualization such as presented in figures 1 and 2 and data analysis of such simulations of finite size objects are expected to give us a new understanding of the dynamics and statistics of material objects in turbulence in terms of the dynamical and statistical properties of the small-scale structures. This kind of study is in progress, and will be reported elsewhere in the near future.

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